Probability and analysis for a hyperbolic coupled PDE system

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Summary

This thesis concentrates on a particular first-order coupled PDE system. It provides both a detailed treatment of the *existence* and *uniqueness* of monotone travelling waves to various equilibria, by differential-equation theory and by probability theory and a treatment of the corresponding hyperbolic initial-value problem, by analytic methods. Numerical techniques are also used, again both from probabilistic and analytic standpoints.

The initial-value problem is studied using characteristics to show existence and uniqueness of a bounded solution for bounded initial data (subject to certain smoothness conditions). The concept of *weak* solutions to partial differential equations is used to rigorously examine bounded initial data with jump discontinuities.

For the travelling wave problem the differential-equation treatment makes use of a shooting argument and explicit calculations of the eigenvectors of stability matrices.

The probabilistic treatment is careful in its proofs of *martingale* (as opposed to merely local-martingale) properties. A modern *change-of-measure* technique is used to obtain the best lower bound on the speed of the monotone travelling wave — with Heaviside initial conditions the solution converges to an approximate travelling wave of that speed (the solution tends to one ahead of the wave-front and to zero behind it). Waves to different equilibria are shown to be related by Doob h-transforms. *Large-deviation theory* provides heuristic links between alternative descriptions of minimum wave speeds, rigorous algebraic proofs of which are provided.

The numerical work concentrates on the initial value problem. Finite difference methods are used to approximate the differential equation (which is most effective via use of the characteristics). The probabilistic system is simulated to obtain an alternative approximation to the solution to the initial value problem using the probabilistic representation.

Contents

Chapter 1

Introduction

1.1 Probability and Differential Equations

In 1937 Fisher [31] introduced the following nonlinear reaction-diffusion equation (in this section we scale out parameters for simplicity of exposition) as a deterministic version of a stochastic model for the spatial spread of a genetic trait:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{1.1}$$

The discussion in Murray [52] develops the background to this area with clear example calculations (further discussion and bibliography are available in many books and review papers, for example the books of Fife [29] and Britton [10]).

Equation (1.1) is the simplest case of a nonlinear reaction-diffusion equation and is the natural extension of the logistic model for population growth,

$$\frac{\partial u}{\partial t} = u(1-u),$$

to incorporate dispersal of the population through space.

Also in 1937, Kolmogorov, Petrovskii and Piskunov [41] studied a similar equation interpreting $u(t,x)$ as the density of some matter that grows and disperses. However there is another, quite different, stochastic interpretation of equation (1.1) introduced in 1975 by a seminal paper of McKean [49]. (Note that this paper is best read with reference to the corrections in McKean [50] and the discussion by Ellis in the review [26].) In the McKean representation $u(t,x)$ is the cumulative distribution function of the right-most particle in a branching Brownian motion (see section 1.3).

These two interpretations correspond to *forward-* and *backward-*equations, the difference in interpretation being most simply illustrated by considering the heat equation, that is, the diffusion part of equation (1.1) without the growth,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{1.2}$$

In the forward sense we would investigate delta-function (at the origin) initial conditions. Then the solution of equation (1.2) is the probability density function of a normal random variable with mean zero and variance increasing linearly with time. In the backward sense we would use Heaviside initial data,

$$u(0,x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{1.3}$$

For Heaviside initial data the solution of equation (1.2) is the cumulative distribution function of the same normal random variable — mean zero and variance increasing linearly with time. Both of these can be interpreted in terms of a particle executing a Brownian motion, starting at the origin. The solution of the forward equation is the probability density function of the particle's position, while the solution of the backward equation is the cumulative distribution function of the particle's position — which is the function describing the probability that the particle is to the left of points as opposed to the probability (heuristically speaking) of being at points.

It is very important to keep clear the distinction between the two interpretations since, for more complicated, less symmetrical situations, the forward and backward equations are different. The model we consider in this thesis is a case in point. We use a backward equation and the loss of symmetry in our system implies that the probabilistic representation of the solution for Heaviside initial data as defined in (1.3) is based on the left-most rather than right-most particle. However, were we to consider data such as

$$u(0,x) = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x \leq 0, \end{cases}$$

the representation would simplify to an expression involving only the right-most particle.

In a general case of a process X_t with generator $\mathcal{G}$ (so that

$$H = h(t,X_t) - h(0,X_0) - \int_0^t \left(\frac{\partial}{\partial t} + \mathcal{G}\right) h(s,X_s)ds$$

is a local martingale) we are interested in the equation

$$\frac{\partial u}{\partial t} = \mathcal{G}u, \quad u(0,x) = f(x). \tag{1.4}$$

The probabilistic representation of the solution of equation (1.4) (also see section 1.3) is

$$u(t,x) = \mathbb{E}^x f(X_t), \tag{1.5}$$

that is, the expectation after time t if the process started at a point x. This may be proved using the martingale $M(s) = u(t-s, X_s)$, which can be shown to be a martingale by Itô's formula. Time can be seen as running backwards in the $t-s$ component of the definition of M. By the martingale property of M, $\mathbb{E}M(t) = \mathbb{E}M(0)$ which is equation (1.5). The fundamental point here is that the x-variable in backward equations corresponds to the starting point of the process, whereas the solution of the forward equation represents the chance of being at point x starting from some fixed point. By space-homogeneity and symmetry equation (1.5) simplifies for branching Brownian motion, see section 1.3.

The forward equation arises directly from consideration of the local martingale H. When h has compact support in $(0,\infty) \times S$ (where S is the state-space of X, and X has density ρ) then the terms outside the integral in the definition of H vanish and therefore (modulo technicalities as to whether H is a true martingale, which we ignore here, but deal with in detail in Chapter 6)

$$0 = \mathbb{E}\int_0^\infty \left(\frac{\partial}{\partial t} + \mathcal{G}\right) h(t, X_t)dt$$

$$= \int_S \int_{t=0}^\infty \left(-\frac{\partial}{\partial t} + \mathcal{G}^*\right) \rho(t,y)h(t,y)dtdy$$

using integration by parts and the density ρ weakly satisfies a differential equation which is a forward equation (the space argument y of a density function $\rho(t,y)$ indicates the position after time t).

Returning to the chronological development of the subject, in 1951 Skellam [65] provided a comprehensive discussion of population growth and spread, putting the biological applications of this theory on a firm footing. Despite the fact that a Brownian path is nowhere differentiable and the fact that a Brownian particle can reach (albeit with very small probability) an arbitrarily distant point in a finite time — which hardly appears to describe processes occurring in the natural world — this modelling continues to find very wide and successful application (the discussion in Skellam [66], for example, demonstrates how good an approximation diffusion can be despite these problems).

The concern over models that postulate inertialess particles with unbounded speed is one of the motivations for the study of correlated random walks, first analysed by Goldstein in 1951 [32]. Rather than the simple, symmetric random walk, from which Brownian motion arises, Goldstein considered a random walk where the increments are not independent, but depend on the direction of the last movement. In the limit this random walk becomes a continuous-time process which changes direction after exponentially distributed time intervals and satisfies the telegraph or telegrapher's equation. This equation, in its simplest form, is as follows:

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

It is such a correlated random walk that underlies the processes studied in this thesis. The historical development of telegraph models is discussed in section 1.4.

We will now review previous contributions in this general area to provide context for our studies.

1.2 Reaction-Diffusion Models

Equation (1.1) is a specific example of the following general nonlinear reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + F(u). \tag{1.6}$$

Here $F(u)$ is the forcing term in the system, for example it is the population growth law, and is such that $F \in C^1[0,1]$ and $F(0) = F(1) = 0$ (hence $u \equiv 0$ and $u \equiv 1$ are particular solutions of equation (1.6)).

A great deal of work has been published on the equation (1.6), under various hypotheses on F. A major aim has been to study whether $u(t,x)$ tends to a travelling front solution as $t \to \infty$. As defined in Fife and McLeod [30], a travelling front is a solution of equation (1.6) of the form $w = u(x - ct)$ for some c (the speed of the front), with the limits $w(\pm\infty)$ existing and unequal; as in Fife and McLeod [30] we consider $w(-\infty) = 0$ and $w(+\infty) = 1$ for definiteness.

Fisher's genetic model [31] corresponded to a special class of F (see Hadeler and Rothe [34]),

$$F(u) = u(1-u)\big(1 - \tau - (2 - \sigma - \tau)u\big), \quad \sigma \geq \tau \geq 0,$$

where σ is the viability of the new (superior) homozygote (an individual with two copies of the new gene), τ the viability of inferior homozygote (an individual with two copies of the old gene) and the viability of the heterozygote (an individual with one copy of each gene) has been scaled to be 1. If the genes have no effect on viability then $\sigma = \tau = 1$ so that $F(u) = 0$ — the new gene simply diffuses.

For $\sigma \geq 1 \geq \tau, \sigma \neq \tau$, the *heterozygote intermediate* case, $F(u) > 0$ for $u \in (0,1)$ (this includes the case $F(u) = u(1-u)$ which corresponds to $\sigma = 2$ and $\tau = 0$). For $\sigma \geq \tau > 1$, the *heterozygote inferior* case, there exists an $\alpha \in (0,1)$ such that

$$F(u) \begin{cases} < 0 & \text{if } u \in (0,\alpha), \\ = 0 & \text{if } u = \alpha \text{ and} \\ > 0 & \text{if } u \in (\alpha,1), \end{cases} \tag{1.7}$$

$$F'(0) < 0, \quad F'(\alpha) \geq 0 \quad \text{and} \quad F'(1) < 0.$$

Conversely if $1 > \sigma \geq \tau$, the *heterozygote superior* case then there exists an $\alpha \in (0,1)$ such that

$$F(u) \begin{cases} > 0 & \text{if } u \in (0,\alpha), \\ = 0 & \text{if } u = \alpha \text{ and} \\ < 0 & \text{if } u \in (\alpha,1), \end{cases} \tag{1.8}$$

$$F'(0) > 0, \quad F'(\alpha) \leq 0 \quad \text{and} \quad F'(1) > 0.$$

Aronson and Weinberger [1] call the case of general F satisfying equation (1.7) heterozygote inferior, and the case of general F satisfying equation (1.8) heterozygote superior, as a natural extension of Fisher's $F(u)$ — terminology which we will also adopt. We will term the case in which $F(u) > 0$ for $u \in (0,1)$ and $F'(0) > 0, F'(0) < 0$ with $F'(0) \geq F'(u)$ for $u \in (0,1)$ the heterozygote intermediate case.

Kolmogorov, Petrovskii and Piskunov [41] studied the heterozygote intermediate case, showing that for Heaviside initial data as in equation (1.3) the solution of the initial value problem *tends* to a travelling front. Precisely stated, there exists a travelling front $w(x-ct)$ and a function $\psi(t)$ (the *correction term*) such that, as $t \to \infty$,

$$u(t,x) - w\big(x - ct - \psi(t)\big) \to 0 \quad \text{uniformly in } x,$$

and $\psi'(t) \to 0$. McKean [49], Larson [44] and Bramson [8, 9] have since improved on Kolmogorov et al.'s estimates of $\psi(t)$ (the probabilistic work is reviewed in more detail in section 1.3). Kolmogorov et al. also proved that there exists a finite speed c_0 such that there exists a travelling front of speed c if, and only if, $c \geq c_0$. The condition $F'(0) \geq F'(u)$ for $u \in (0,1)$ allowed them to calculate c_0 explicitly. Hadeler and Rothe [34] (building on work in Rothe's thesis [63]) extended these results to the heterozygote inferior case and calculated the minimal wave speed even when $F'(u)$ does not attain its maximum at 0.

Fife and McLeod [30] proved results about uniform convergence to certain travelling front configurations in the heterozygote superior case, and generalizations (additional zeroes of F between 0 and 1) of it, and extend results of Aronson and Weinberger [1] on stability to perturbations of compact support (when the disturbances are large they propagate, at approximately constant speeds).

Chauvin [14, 15], Chauvin and Rouault [16, 17], Lalley and Sellke [42] and Neveu [53] provided an important completion of McKean's work in [49], dealing probabilistically with the elusive case of waves travelling at the minimal wave speed. These papers made important use of *multiplicative* martingales, as opposed to the additive Z_b martingales used by McKean [49, 50] (see section 1.3), though the multiplicative martingales are implicit in the McKean representation. These martingales are connected through the ideas of stopping lines and branching trees.

At this stage it is appropriate to discuss McKean's work in more detail.

1.3 McKean's representation

Firstly note that McKean [49, 50] transformed equation (1.1) into *backward* form by the substitution $u \to 1-u$, giving the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u. \tag{1.9}$$

McKean made use of a branching Brownian motion X. His description of that process is as follows. At time 0 a single particle commences a Brownian motion X_1 on $\mathbb{R}$, starting from the origin. After an exponentially distributed (with parameter 1) time T independent of X_1, that is with $\mathbb{P}(T > t) = e^{-t}$, the particle splits into two, the new particles executing independent

Brownian motions starting from $X_1(T)$. These particles also split after independent exponential times, as do their descendants ad infinitum. Thus, after an elapsed time $t > 0$, there are $N(t)$ particles located at positions $X_1(t), X_2(t), \dots, X_{N(t)}(t)$, and the state of the system is

$$X(t) = \big(X_1(t), X_2(t), \dots, X_{N(t)}(t)\big).$$

If the initial data $u(0,x) = f(x)$ (for a function f bounded between 0 and 1) then the McKean representation of the solution $u(t,x)$ of equation (1.9) is simply

$$u(t,x) = \mathbb{E} \prod_{k=1}^{N(t)} f(x + X_k).$$

In the case of Heaviside initial data, defined in (1.3), this simplifies to

$$u(t,x) = \mathbb{P}\left[\min_{k \le N(t)} X_k(t) + x > 0\right].$$

As McKean points out, due to the symmetry of Brownian motion, this is equal to

$$\mathbb{P}\left[\max_{k \le N(t)} X_k(t) < x\right]$$

(which is the cumulative distribution function of the right-most particle).

McKean [49, 50] links the following non-negative martingale $Z_b(t)$ (first discovered by Watanabe [73], also see related work on use of martingales in branching processes by Biggins [5, 6, 7] and Uchiyama [69, 70]) with the travelling-wave solutions. We set

$$Z_b(t) = e^{-t} \sum_{k=1}^{N(t)} \exp\{-bX_k(t) - \frac{1}{2}b^2 t\}.$$

Note that $Z_b(\infty)$ exists almost surely because Z_b is a non-negative martingale. For $0 < b < \sqrt{2}$, Z_b converges to a non-zero limit $Z_b(\infty)$ and defining $w_c(x)$ by

$$w_c(x) = \mathbb{E}^x \exp[-Z_b(\infty)]$$

gives a travelling-wave solution of equation (1.9) of speed

$$c = \frac{1}{b} + \frac{b}{2} > \sqrt{2}.$$

The critical case ($b = \sqrt{2}$) was completed by Neveu [53] with the following result. Define another martingale $V(t)$ by

$$V(t) = -(\partial/\partial b) Z_b(t)|_{b=\sqrt{2}}.$$

This martingale converges to a non-negative limit $V(\infty)$ not in $L^1(\mathbb{P})$ and

$$w_{\sqrt{2}}(x) = \mathbb{E}\exp\{-e^{-x\sqrt{2}}V(\infty)\}$$

gives the unique (modulo translation) travelling-wave solution of equation (1.9) of speed $\sqrt{2}$.

The probabilistic method was also used by Bramson [8] who showed that the correction term $\psi(t) = \frac{3}{2\sqrt{2}}\log t + O(1)$ (see also Bramson [9]).

Champneys, Harris, Toland, Warren and Williams [13] study a generalized branching Brownian motion which has two *types*, further generalizations can also be seen in Harris [35], Harris and Williams [36] and Warren [72]. A feature of this work is the use of large-deviation theory (see, for example, Ellis [27]) on the occupation times of the types and this idea is made use of in this thesis.

1.4 Telegraph Models

We follow the description of a correlated random walk in Othmer, Dunbar and Alt [58] (which Protopescu and Keyes [60] call the Goldstein-McKean model after the papers of Goldstein [32] and McKean [48]).

Suppose that a particle moves along the x-axis at a constant speed b, but that at random instants of time it reverses direction. Suppose that this reversal process is a Poisson process with intensity θ, that is, the rate of reversal per unit time is θ. Let $u_1(t,x)$ be the probability density of particles that are at (t,x) and are moving to the right and let $u_2(t,x)$ be the probability density of particles that are at (t,x) and are moving to the left. Then u_1 and u_2 satisfy the equations

$$\frac{\partial u_1}{\partial t} + b\frac{\partial u_1}{\partial x} = \theta(u_2 - u_1), \quad (1.10)$$

$$\frac{\partial u_2}{\partial t} - b\frac{\partial u_2}{\partial x} = \theta(u_1 - u_2). \quad (1.11)$$

The probability that a particle is at (t,x) is $p(t,x) = u_1(t,x) + u_2(t,x)$ and we define the probability flux to be $j = b(u_1 - u_2)$. The connections between this model and hydrodynamic flows (hence the indicative term flux) are discussed in Protopescu and Keyes [60] and links with Kirchoff's laws are explored probabilistically in Orsingher [57]. The flux j and the probability p satisfy the following equations (by adding equation (1.10) to equation (1.11) to get equation (1.12) and subtracting b times equation (1.11) from b times equation (1.10) to get equation (1.13))

$$\frac{\partial p}{\partial t} + \frac{\partial j}{\partial x} = 0, \quad (1.12)$$

$$\frac{\partial j}{\partial t} + b^2\frac{\partial p}{\partial x} = -2\theta j. \quad (1.13)$$

The system (1.12), (1.13) is equivalent to the telegraph equation with parameters as shown below (which can be seen by differentiating with respect to t in equation (1.12), with respect to x in equation (1.13) and then substituting for all terms in j)

$$\frac{\partial^2 p}{\partial t^2} + 2\theta \frac{\partial p}{\partial t} = b^2 \frac{\partial^2 p}{\partial x^2}.$$

In fact we could get an equation in identical form for j by different differentiation and substitution — the only difference being in the initial conditions.

This model has attracted interest in a wide variety of fields. For example Levin [46] and Okubo [54] both mention its use in ecological modelling, Weiss and Rubin [74] include it in their review of random walks for physical chemistry (it has obvious links with discrete-velocity models for gases and other transport problems, see the review article by Platkowski and Illner [59]) and Bartlett [2] makes connections with the Schrödinger equation. This model is discussed, and the telegraph equation derived, in the context of random evolutions, by Ethier and Kurtz [28, pages 468–470].

The model has been generalized in a number of ways. Bartlett [3] considers allowing the rate of reversals to depend on the direction of travel and even be inhomogeneous in space. He also discusses some extensions to more space dimensions, though as he says it is less clear for correlated random walks, as opposed to simple symmetrical random walks, how this should be done.

Orsingher [55, 56] extended the model in two dimensions and connects it to wave equations and laws of electromagnetics, following work of Cane [11] on diffusion models with relativity effects. Indeed in [56] Orsingher states that "A parallel aim of this paper is to show that all basic linear equations of mathematical physics can be derived on the basis of stochastic reasoning". While this is certainly an interesting goal it is the non-linear generalizations of the model which interest us here.

It is surprising to note that the addition of *reaction* terms to these telegraph models appears not to have been done until the last 10 years – Holmes [38] coins the phrase reaction-telegraph models in an interesting comparison of reaction-diffusion models and reaction-telegraph models for biologically realistic parameter values. For these parameter ranges the differences in model predictions are surprisingly small. Holmes uses a forward version of the equations where the solutions represent the density of the particles. Her equations also retained symmetry, both in rate of reversals and in growth rates, for particles with either velocity.

On the other hand Dunbar [24] added non-linearity and used a McKean representation, the solution being the cumulative distribution function of the extremal particle. However, again the model investigated was symmetrical between particles of either velocity.

Hadeler [33] generalized the form of the non-linearity, much in the way that Hadeler and Rothe [34] and Fife and McLeod [30] generalized the non-linearity for reaction-diffusion systems.

We break the symmetries used by the above authors which necessitates some different techniques in the analysis. For example it is no longer possible to simplify the equations by addition and subtraction in a straightforward manner to get total probability and flux as our two components. However this actually proves advantageous because maintenance of the obvious (from the stochastic viewpoint) original pair of first order PDEs naturally leads us to use the method of characteristics. This enables us to develop existence and uniqueness theory even for certain weak solutions (using a result of Beale [4] developed in discrete-velocity gas modelling) and also yields comparison results. We also introduce some new uses of large-deviation and numerical techniques to this problem. Analytically this model proves particularly interesting because it exhibits persistent discontinuities while remaining amenable to explicit calculations. The method used in Chapter 7 deals with a model which has $n \geq 2$ states which corresponds to n coupled first order PDEs or a single n-th order PDE.

Chapter 2

The problem

2.1 A coupled equation system

Let r_1, r_2, q_1, q_2 be positive constants and b_1, b_2 unrestricted real constants, fixed throughout. Let θ be a *positive* rate parameter — the larger this parameter is, the faster certain things happen (so it can be thought of as analogous to temperature). We consider an equation system related to the generalized FKPP system discussed in Champneys, Harris, Toland, Warren and Williams [13] (which contains further references for this area). Here the system of interest is

$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial x} + R(u^2 - u) + \theta Q u, \tag{2.1}$$

where u is a vector-valued function from $[0,\infty) \times \mathbb{R}$ to $\mathbb{R}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $u^2 = (u_1^2, u_2^2)$, and where

$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad R := \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad Q := \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

(We use ':=' to mean 'is defined to equal'.)

Dunbar [24] considered a similar system but with $q_1 = q_2$, $r_1 = r_2$ and the nonlinearity was $\frac{1}{4}(u_1 + u_2)^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ rather than u^2. Holmes [38] compared reaction-diffusion systems to reaction-telegraph models (again with symmetry in the nonlinearity in both terms) for animal movement; if equation (2.1) is rewritten as a single second-order PDE it can be seen that our system is a reaction-telegraph model, though we find it more convenient to work with the pair of first-order equations. Hadeler [33] discussed more general non-linearities (though not the one we study), but still retained the condition $q_1 = q_2$. We contrast the probabilistic interpretation of these models and ours in section 2.5.

A travelling-wave solution of equation (2.1) is a solution of the form $u(t, x) := w(x - ct)$ (where $w : \mathbb{R} \to \mathbb{R}^2$). w describes a travelling wave if and only if

$$(B + cI)w' + R(w^2 - w) + \theta Q w = 0. \tag{2.2}$$

This equation has equilibria at points where the 2-dimensional vector w satisfies

$$R(w^2 - w) + \theta Q w = 0.$$

The '*source point*' $S = (0,0)$ and the '*target point*' $T = (1,1)$ are clearly equilibria, since Q has zero row-sums. If $r_1 r_2 \geq 4\theta^2 q_1 q_2$, there will also be equilibria at the two points

$$E_{\pm} = \left(\frac{1}{2} + \theta\rho_1 \pm \sqrt{\Delta}, \frac{1}{2} + \theta\rho_2 \mp \sqrt{\Delta}\right),$$

where

$$\rho_i := q_i / r_i \text{ and } \Delta := \frac{1}{4} - \theta^2 \rho_1 \rho_2.$$

For $\theta \in (0, \infty)$ we study the existence of monotone travelling waves from S to T, that is, solutions of equation (2.2) for which both components of w are increasing functions that go from 0 to 1 as the argument goes from $-\infty$ to $+\infty$. Waves from S to the other equilibria can be obtained through a transformation detailed in section 2.4.

2.2 Stability of equilibria

Suppose that $(B + cI)$ is invertible (i.e. $c \notin \{-b_1, -b_2\}$) and write equation (2.2) in the form

$$\frac{\mathrm{d}w}{\mathrm{d}x} = F(w). \tag{2.3}$$

Then F is a quadratic polynomial. Let E be an equilibrium point of (2.2), thus $F(E) = 0$. Then write $w(x) - E = v(x)$ and expand equation (2.3) to first order in v. This yields

$$\frac{\mathrm{d}v}{\mathrm{d}x} = K(E)v, \qquad K_{ij}(E) := \frac{\partial F_i}{\partial w_j} \text{ evaluated at } w = E.$$

The matrix $K(E)$ is called the *stability matrix* at E of equation (2.3). The dimension of the *stable* [respectively, *unstable*] *manifold* of (2.3) at E is the number of eigenvalues of $K(E)$ of *negative* [respectively, *positive*] real part, counting algebraic multiplicity. See Carr [12], Coddington and Levinson [18] and Hartman [37].

Thus $K_{c,\theta}(T)$ (we write $K_{c,\theta}$ to emphasise the dependence on c and θ) satisfies

$$(B + cI)K_{c,\theta}(T) + R + \theta Q = 0 \tag{2.4}$$

and $K_{c,\theta}(S)$ satisfies

$$(B + cI)K_{c,\theta}(S) - R + \theta Q = 0.$$

The stability properties of these matrices are investigated in Chapter 4.

2.3 The main ODE theorem

DEFINITION. An eigenvalue λ of a real 2×2 matrix M will be called *stable* [respectively, *unstable*] *monotone* if

(i) λ is real and negative [positive], and

(ii) M has an eigenvector (v_1, v_2) corresponding to λ with $v_1 v_2 \geq 0$.

This definition links nicely with the Perron-Frobenius Theorem (see, for example, the book by Seneta [64]). The theorem implies that a square matrix M with all off-diagonal entries strictly positive has a special eigenvalue $\Lambda_{PF}(M)$ with an associated eigenvector with all entries positive and such that every other eigenvalue of M has real part less than $\Lambda_{PF}(M)$. Moreover, any eigenvector with all entries positive must be a multiple of the Perron-Frobenius eigenvector.

As will be proven in Chapter 4 (Lemma 4.1), for any fixed $\theta > 0$, there exists a critical value $c(\theta)$ in the interval $\min(-b_1, -b_2) \leq c(\theta) \leq \max(-b_1, -b_2)$ — with the property that if $c > c(\theta)$, $c \neq \max(-b_1, -b_2)$, then $K_{c,\theta}(T)$ has at least one stable monotone eigenvalue, and if $c < c(\theta)$, $c \neq \min(-b_1, -b_2)$, it has no such eigenvalues. Since a necessary condition for the existence of a monotone travelling wave of (2.2) which converges to T as $x \to \infty$ is the existence of a stable monotone eigenvalue of $K_{c,\theta}(T)$ (for convergence to an equilibrium a necessary condition is an eigenvalue with non-positive real part — but for this system this means that the eigenvalue must in fact be negative, as a zero eigenvalue is not possible; for this convergence to be monotone it is necessary that the eigenvalue is real and a corresponding eigenvector has both components of the same sign) this provides a lower bound on possible values of c for which a monotone travelling wave can exist. However, it is shown later that this condition on c is sufficient for the existence of a monotone connection from S to T. Large-deviation theory gives probabilistic heuristics for this critical value, see section 6.3. Thus we can state our main result as follows.

Theorem 2.1 *Suppose that $c > c(\theta)$, then there exists one and, modulo translation, only one monotone solution of equation* (2.2) *with $w(x) \to S$ as $x \to -\infty$ and $w(x) \to T$ as $x \to +\infty$. For $c < c(\theta)$ there is no such solution.*

This theorem is proven in Chapter 5 using a shooting argument. Behaviour at $c(\theta)$ itself depends on the relative values of parameters as follows (these special cases are detailed in section 5.6):

- If $c(\theta)$ is in the interval $\min(-b_1, -b_2) < c(\theta) < \max(-b_1, -b_2)$ then there is a unique, monotone solution for $c = c(\theta)$;
- For $i = 1, 2$, if $c(\theta) = -b_i$, then there is a unique monotone solution for $c = c(\theta)$ if and only if $\theta q_i = r_i$.

2.4 A Doob h-transform

As in Champneys et al. [13] there is a *Doob h-transform* that maps monotone waves from S to one of the equilibria to monotone waves from S to another of the equilibria. This fact is summarised by the following Lemma. The probabilistic meaning of this is discussed in section 6.8.

Lemma 2.2 *Suppose that θ is fixed at a value where $\Delta \geq 0$, so that E_+ and E_- exist. If $E = (\alpha_1, \alpha_2)$ denotes either E_+ or E_-, then the substitution*

$$\tilde{q}_i := q_i \alpha_j / \alpha_i \quad (j \neq i), \quad \tilde{r}_i := r_i \alpha_i, \quad \tilde{u}_i := u_i / \alpha_i, \quad \tilde{w}_i := w_i / \alpha_i \tag{2.5}$$

transforms (2.1) *and* (2.2) *into their* $\sim$ *versions, monotone waves from S to E for the original problem corresponding exactly to monotone waves from S to T for the* $\sim$ *problem. The possibility that $E_+ = E_-$ and $\Delta = 0$ is not excluded.*

Much of our work therefore automatically transfers to the case when T is replaced by E_+ or E_-, though the critical values $c^\pm(\theta)$ of θ corresponding to waves from S to $E^\pm$ will be different.

2.5 Key probability theorems

Our aim in Chapter 6 will be to prove the following theorems which connect a probability model, defined below, with the system (2.1). Note that we often switch between two equivalent notations in Chapter 6 to reduce the use of subscripts. Thus we will sometimes write, for example, $b(y)$ for b_y, $w(x, y)$ for $w_y(x)$ and $u(t, x, y)$ for $u_y(t, x)$ (for $y = 1, 2$).

Let $I := \{1, 2\}$ and consider the following two-type branching system of particles. At time $t \geq 0$, there are $N(t)$ particles, the k-th particle — *in order of birth* — having *position* $X_k(t)$ in $\mathbb{R}$ and *type* $Y_k(t)$ in I. The *state* of the system at time t is therefore

$$\Big(N(t);\, X_1(t), \ldots, X_{N(t)}(t);\, Y_1(t), \ldots, Y_{N(t)}(t)\Big). \tag{2.6}$$

Particles, once born, behave independently of one another. Each particle lives for ever. The type of a particle (once born) is an autonomous Markov chain on I with Q-matrix θQ. While a particle is of type $y \in I$, it moves with constant velocity $b(y)$, and it gives birth — *to one child each time, at its own current position and of its own current type* — in a Poisson process of rate $r(y)$. So, $r(y)$ is the breeding rate of type y.

The branching system in Dunbar [24] had $q_1 = q_2$, $r(1) = r(2)$ and particles, rather than giving birth to one particle of the same type and living on themselves, die and give birth to two particles of independent *random* types, with each having equal probability of being of either type. The equations studied by Hadeler [33] correspond to the new pair of particles having type correlated to that of their parents, for any correlation except ± 1.

For our model it makes no difference whether you consider that one new particle has been born and the old one lives on too, or that two new particles of the same type are born. However, when the type of the new particles is random, this distinction is crucial. Our model is thus distinct from those previously considered.

Write $\mathbb{P}_{x,y}$ (with associated expectation $\mathbb{E}_{x,y}$) for the law of this process when it starts from one particle of type $Y_1(0) = y$ at position $X_1(0) = x$. By martingale [respectively, local martingale, supermartingale, ...] we mean a process which is for every $\mathbb{P}_{x,y}$ a martingale [respectively, ...] relative to the natural filtration $\mathcal{F}_t$ ($\mathbb{P}_{x,y}$-augmented, to be precise) of the process at (2.6).

The state-space for this process is

$$\mathcal{S} := \bigcup_{n\geq 1}\Big(\{n\}\times\mathbb{R}^n\times I^n\Big). \tag{2.7}$$

Define $L(t) := \inf_{k\leq N(t)} X_k(t)$. This is the position of the left-most particle. The asymptotic speed of the left-most particle is $\lim_{t\to\infty} t^{-1}L(t)$.

Theorem 2.3 *As $t\to\infty$, the following holds almost surely* (a.s.)

$$t^{-1}L(t) \to -c(\theta). \tag{2.8}$$

If u satisfies the coupled system (2.1)*, for $t\geq 0$ and $x\in\mathbb{R}$ and if*

$$u(0,x,y) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

then for $t > 0$, $0 \leq u(t,x,y) \leq 1$, $u(t,x,y) = \mathbb{P}_{x,y}[L(t) > 0]$, and u is an approximate travelling wave of speed $c(\theta)$ in the sense that

$$u(t,x+\gamma t,y) \to \begin{cases} 0 & \text{if } \gamma < c(\theta), \\ 1 & \text{if } \gamma > c(\theta). \end{cases}$$

This theorem is proved, along with the following one, in sections 6.5 and 6.6. The theorem allows us to relate the speed of the spread of the particles in the probabilistic model with the wave speed of travelling waves. Specifically, we are claiming that the left-most particle travels, in the limit, at the speed $-c(\theta)$, where $c(\theta)$ is the critical speed above which a unique monotone travelling wave exists and below which no such wave exists.

An analytic proof that the weak solution of (2.1) is between 0 and 1 for the Heaviside initial data is included in section 3.6, as well as a proof that for continuous initial data between 0 and 1 the solution remains between 0 and 1 (see Lemma 3.2 and the subsequent remarks). The work of Chapter 3 is not necessary for the probabilistic approach to the problem but adds insight from the viewpoint of classical analysis, and vice-versa. In Chapter 6 we show that any (smooth) solution to the coupled system (2.1) that is between 0 and 1 has a McKean representation. We use this representation to motivate the (probabilistic) construction of a solution for the initial-value problem with Heaviside initial data. This constructed solution is then directly verified to satisfy the appropriate equations and does remain between 0 and 1.

Chapter 7 introduces a new approach to obtaining the McKean representation, avoiding the use of formal generators and extending the representation to non-smooth initial data and *weak* solutions of (2.1). With use of a simple scaling transformation the McKean representation can also be extended outside the interval $[0, 1]$, this is discussed in section 9.4.

Consider the case when $X_1(0) = 0$ and $Y_1(0) = 1$, that is, work with the $\mathbb{P}_{0,1}$ law: $\mathbb{P} := \mathbb{P}_{0,1}$. The terminology — *probabilistic* eigenvalue of $K_{c,\theta}(T)$ — is a shorthand explained fully after Theorem 6.2.

Theorem 2.4 (i) *Let $c > c(\theta)$. Let λ be the* probabilistic *eigenvalue of $K_{c,\theta}(T)$. Define*

$$Z_\lambda(t) := \sum_{k=1}^{N(t)} v_\lambda(Y_k(t)) \exp\Big\{ \lambda[X_k(t) + ct] \Big\},$$

with v_λ being the eigenvector (with $v_\lambda(1) = 1$) corresponding to $\Lambda_{PF}(\lambda)$, the Perron-Frobenius eigenvalue of $\lambda B + \theta Q + R$. The fact that Z_λ is a martingale (see the discussion after Theorem 6.2) implies that

$$\liminf_{t\to\infty} t^{-1} L(t) \geq \lambda^{-1}\Lambda_{PF}(\lambda) \quad \text{(a.s.)}.$$

(ii) *Since $Z_\lambda(\infty)$ exists in $\mathcal{L}^1$ (by Theorem 6.4) and $Z_\lambda(0) = 1$, we can define a measure Q_λ equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty$ by*

$$dQ_\lambda/d\mathbb{P} = Z_\lambda(\infty) \ \ on\ \mathcal{F}_\infty, \ \ whence\ dQ_\lambda/d\mathbb{P} = Z_\lambda(t)\ \ on\ \mathcal{F}_t.$$

Then

$$M_\lambda(t) := Z_\lambda(t)^{-1} \frac{\partial}{\partial\lambda} Z_\lambda(t)$$

defines a Q_λ-martingale, and

$$t^{-1} M_\lambda(t) \to 0 \quad \text{(a.s.)}.$$

This implies that

$$\limsup_{t\to\infty} t^{-1} L(t) \leq \frac{\partial}{\partial\lambda} \Lambda_{PF}(\lambda) \quad \text{(a.s.)}.$$

(iii) *As $c \downarrow c(\theta)$, we have*

$$\lambda^{-1}\Lambda_{PF}(\lambda) \to -c(\theta) \quad \text{and} \quad \frac{\partial}{\partial\lambda}\Lambda_{PF}(\lambda) \to -c(\theta),$$

so that (2.8) *follows.*

This result gives us the weaponry to prove Theorem 2.3 using martingale techniques. Note that Theorem 2.3 is proved using Theorem 2.4, and that in section 6.5 we prove Theorem 2.4 via the Theorems and Lemmas in the preceding sections of Chapter 6 — hence various references to Theorems of Chapter 6 in the statement of Theorem 2.4 are not circular.

2.6 Numerical work

Numerical solutions of the system (2.1) are presented and discussed in Chapters 8 and 9. Chapter 8 compares and contrasts the best finite difference methods with plots obtained via probabilistic simulation (for Heaviside initial data) and gives sample code for both approaches. Chapter 9 covers the other methods tested and presents examples for alternative initial data, as well as an examination of the way in which numerical solutions converge to travelling waves.

2.7 Summary chart of results

This table is intended to help keep track of the various cases determined by the values of parameters. There is no monotone travelling wave from S to T if No appears in both right-hand columns. The inside and outside regions mentioned here are defined in Chapter 5 where it is shown that if a monotone travelling wave exists then it lies either in an inside, or an outside region.

	Number of unstable *monotone* eigenvalues at S	Number of stable *monotone* eigenvalues at T	Monotone Connection through	
			inside region	outside region
$c > \max(-b_1, -b_2)$	1	1	Yes	No
$\max(-b_1, -b_2) > c > c(\theta)$	1	2	No	Yes
$c(\theta) > c > \min(-b_1, -b_2)$	1	0	No	No
$c < \min(-b_1, -b_2)$	0	0	No	No

When $c \in \{-b_1, -b_2\}$ the coupled system (2.1) becomes an ODE and an algebraic equation. The only candidate for a travelling wave from S to T is the corresponding segment of the solution of the algebraic equation (which is a parabola) — full details of this special case are in section 5.6.

Chapter 3

Existence and uniqueness results for the PDE initial-value problem

3.1 Introduction

The system of interest is the semi-linear hyperbolic system of partial differential equations:

$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial x} + R(u^2 - u) + \theta Q u.$$

To study the question of global existence and uniqueness of solutions to the initial-value problem for this system, it is convenient to change to moving coordinates (moving at a speed of $\frac{1}{2}(b_1+b_2)$) and then re-scale space so that the coefficients of u_x are 1 and -1. This is possible unless $b_1 = b_2$ — we deal with this case in section 3.8. We use subscript notation to represent derivatives and relabel so that the system becomes:

$$u_t - u_x = \quad r_1(u^2 - u) + \theta q_1(v - u) =: \quad f(u, v); \tag{3.1}$$

$$v_t + v_x = \quad r_2(v^2 - v) + \theta q_2(u - v) =: \quad g(u, v). \tag{3.2}$$

The functions f and g are introduced to simplify notation.

We are particularly interested in the Cauchy problem for Heaviside initial data:

$$u(0, x) = v(0, x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

which we study in section 3.6, but first we study the Cauchy problem with C^1-initial data.

The characteristics are, using characteristic co-ordinates, $\frac{x+t}{2} = \tau$, $\frac{t-x}{2} = \chi$:

$$\tau \quad = \text{constant, on which} \quad \frac{\partial u}{\partial \chi} = f(u, v);$$

$$\chi \quad = \text{constant, on which} \quad \frac{\partial v}{\partial \tau} = g(u, v).$$

Integrating along the characteristics from $t = 0$ to a point (T, X) gives:

$$\begin{aligned} u(T,X) &= u(0,X+T) + \int_{-\frac{T+X}{2}}^{\frac{T-X}{2}} f((u,v)(\frac{X+T}{2},\chi))d\chi; \\ v(T,X) &= v(0,X-T) + \int_{\frac{X-T}{2}}^{\frac{X+T}{2}} g((u,v)(\tau,\frac{T-X}{2}))d\tau. \end{aligned}$$

We work in the Banach space C^1 of 2-vector-valued functions w for which w and w_x are continuous and bounded for all x, in which we choose as norm $|||w||| = \max(||w||, ||w_x||)$, where $||w|| := \sup_{x\in\mathbb{R}} |w(x)|$, the usual L^∞-norm. These equations can be used with the Contraction Mapping Principle to prove existence and uniqueness of classical solutions over short time (up to time t_0, say) for C^1-initial data. These are functions from $[0, t_0]$ into C^1 such that $w(t, x)$ satisfies equations (3.1) and (3.2) with C^1-initial data. See Courant and Hilbert [20, pages 461–471] and John [39, pages 44–48] for details.

3.2 Proofs for smooth initial data

To go from short-time existence and uniqueness of solutions to global existence and uniqueness we prove the following lemmas which give bounds on the solutions for all time (for a certain class of initial data). These bounds then allow iteration of the Contraction Mapping argument — hence local existence and uniqueness become global. This iteration is done by using the solution obtained from local existence and uniqueness, up until a small, fixed time, t_0, then taking the value of this solution at time t_0 as initial data, and repeating the argument. The bounds obtained in the following two lemmas allow repeated use of the same t_0 at each step, rather than having to take a sequence of t_n (whose sum may converge to a finite *blow-up* time), thus we obtain existence and uniqueness for all time. To set our notation, note that we will write $u(0, x) = u_0(x), v(0, x) = v_0(x)$ for $-\infty < x < \infty$.

Lemma 3.1 *If* $0 < u_0(x), v_0(x) < K \leq 1$ *for all* x*, and* (u_0, v_0) *is in* C^1*, then* C^1 *solutions* (u, v) *of equations* (3.1), (3.2) *satisfy* $0 < u, v < K$ *for all time.*

Proof. Suppose, for a contradiction, that there exists a point (T, X) where (u, v) is outside the square, $(0, K)^2$. The value of (u, v) at this point only depends on the initial data in the interval $[X - T, X + T]$ — this is the domain of dependence (see Courant and Hilbert [20, pages 438–440]). This data determines the solution (u, v) throughout the closed triangle, which we shall denote by Ω, of (t, x)-space whose corners are $(T, X), (0, X - T)$ and $(0, X + T)$.

Since Ω is compact and u and v are continuous, for each of the possible violations (that is, violations of the four inequalities $u < K$, $u > 0$, $v < K$ and $v > 0$) we can find a *first time* it occurs in Ω. For example, if $u \geq K$ at some point in Ω then there exists a point $(t_0, x_0) \in \Omega$ such that $u(t_0, x_0) = K$ and, for all $(t, x) \in \Omega$ with $t < t_0$, $u(t, x) < K$. Similarly we can find a time (and corresponding spatial position) where the first violations of $u > 0$, $v < K$ and $v > 0$ occur in Ω (if such violations do occur). We can then study the *first violation* that happens, by taking the one corresponding to the minimum of these 4 times (taking the time to be $T+1$ if it does not occur in Ω). This is well-defined because we know there is at least 1 violation and at most 4.

So, consider each possible *first violation* in turn. Firstly, that there exists a point $(t_0, x_0) \in \Omega$ such that $u(t_0, x_0) = K$ and, for all $(t, x) \in \Omega$ with $t < t_0$, $\big(u(t,x), v(t,x)\big) \in (0, K)^2$. Then consider u restricted to the characteristic $x = x_0 + t_0 - t$ through (t_0, x_0) (which lies entirely in Ω). From (3.1), with $u = u(t, x_0 + t_0 - t)$,

$$\begin{aligned} \frac{\partial u}{\partial \chi} = r_1(u^2 - u) + \theta q_1(v - u) \quad &< \quad r_1(u^2 - u) + \theta q_1(K - u) \\ &\leq \quad \theta q_1(K - u). \end{aligned}$$

Therefore, $u \leq K + (u_0 - K)\exp(-\theta q_1 \chi) < K$ where $u_0 = u(0, x_0 + t_0)$, for $0 \leq t \leq t_0$, which contradicts $u(t_0, x_0) = K$.

Similarly, we can show that $v(t,x) < K$ for all $(t, x) \in \Omega$, and we use a similar argument to show $u > 0$ and $v > 0$:

$$\frac{\partial u}{\partial \chi} \geq -(r_1 + \theta q_1)u,$$

so $u \geq u_0 \exp\big(-(r_1 + \theta q_1)\chi\big) > 0$ if $u_0 > 0$.

Hence no violation occurs in Ω which is a contradiction. This completes the proof. □

Lemma 3.1 allows us to prove the following:

Lemma 3.2 *If* $0 \leq u_0(x), v_0(x) \leq K$ *for all* x*, for some constant,* $0 < K \leq 1$*, and* u_0, v_0 *are in* C^1*, then there exists a unique (global)* C^1 *solution* (u, v) *of equations* (3.1), (3.2) *satisfying* $0 \leq u, v \leq K$ *for all time.*

Proof. Consider a sequence $(u^n, v^n)(x)$ of initial data satisfying the conditions of Lemma 3.1, which converges in C^1 to $(u_0, v_0)(x)$. That is, each $(u^n, v^n) \in C^1$, $0 < u^n, v^n < K$,

$$\sup_x \left(|u_0 - u^n|, |v_0 - v^n|\right) \to 0 \text{ as } n \to \infty$$

and

$$\sup_x \left(\left|\frac{\partial u_0}{\partial x} - \frac{\partial u^n}{\partial x}\right|, \left|\frac{\partial v_0}{\partial x} - \frac{\partial v^n}{\partial x}\right| \right) \to 0 \text{ as } n \to \infty.$$

Up to any fixed time T, solutions $u^{(n)}(t,x), v^{(n)}(t,x)$ satisfy $0 < u^{(n)}, v^{(n)} < K$ (Lemma 3.1), so that, $\lim_{n\to\infty}(u^{(n)}, v^{(n)}) = (u, v)$ lies in $[0, K]^2$. The fact that this limit exists and is a unique solution is standard (see Courant and Hilbert [20, pages 467–468] and John [39, pages 47–48]). Since T is arbitrary the result is true for all time. □

This lemma completes the proof of global existence and uniqueness of bounded solutions for smooth initial conditions between 0 and any constant $0 < K \leq 1$. This upper limit on K is the best possible, since $u^2 - u > 0$ for $u > 1$, so the solution tends to grow. Indeed, it is clear that for $u_0(x) = v_0(x)$ identically equal to $1 + \epsilon$, for any $\epsilon > 0$, the solution blows up in finite time.

However it is possible to extend the above lemmas to deal with initial data below 0, since the $u^2 - u$ nonlinearity will tend to push the solution up towards 0.

Lemma 3.3 *If $-\infty < K < u_0(x), v_0(x) < 0$ for all x, and (u_0, v_0) is in C^1, then C^1 solutions (u, v) of equations* (3.1), (3.2) *satisfy $K < u, v < 0$ for all time.*

Proof. We can again look for the *first violation*, this time of the four restrictions $u > K, u < 0, v > K, v < 0$, and look at a characteristic going through a point at which the first violation occurs.

For the case $u = K$, note that

$$\begin{aligned} \frac{\partial u}{\partial \chi} = r_1(u^2 - u) + \theta q_1(v - u) &> \theta q_1(v - u) \\ &> \theta q_1(K - u), \end{aligned}$$

so that u does not hit K.

For $u = 0$, note that, looking at the characteristic sufficiently close to the violation point (so that $u > -1$)

$$\begin{aligned} \frac{\partial u}{\partial \chi} = r_1(u^2 - u) + \theta q_1(v - u) &< r_1(-2u) + \theta q_1(-u) \\ &< -(2r_1 + \theta q_1)u, \end{aligned}$$

so that u does not hit 0 either.

Similarly for v. Thus all the contraction mapping arguments extend to all time in the same way that the bounds on data between 0 and K (for $0 < K \leq 1$) extended existence and uniqueness in that case. □

In the same way that we passed from Lemma 3.1 to Lemma 3.2 we can relax the strict inequalities in the hypotheses of Lemma 3.3 to obtain the following result.

Lemma 3.4 *If $-\infty < K \leq u_0(x), v_0(x) \leq 0$ for all x, for some constant $K < 0$, and (u_0, v_0) is in C^1, then C^1 solutions (u, v) of equations* (3.1), (3.2) *satisfy $K \leq u, v \leq 0$ for all time.*

Combining Lemma 3.2 and Lemma 3.4 prepares the ground for the following result.

Lemma 3.5 *If $K_1 < u_0(x), v_0(x) < K_2$ for all x, for some constants $-\infty < K_1 \leq 0$ and $0 \leq K_2 \leq 1$, and (u_0, v_0) is in C^1, then C^1 solutions (u, v) of equations* (3.1), (3.2) *satisfy $K_1 < u, v < K_2$ for all time.*

Proof. If $K_1 = 0$ then this result is simply a restatement of Lemma 3.2, and Lemma 3.4 deals with the case $K_2 = 0$. So, we may assume that $K_1 < 0 < K_2$.

For a violation such as, say, $u = K_1$, note that, considering a section of the characteristic sufficiently close to the violation point for $u < 0$

$$\begin{aligned} \frac{\partial u}{\partial \chi} = r_1(u^2 - u) + \theta q_1(v - u) &> \theta q_1(v - u) \\ &> \theta q_1(K_1 - u), \end{aligned}$$

so that u does not hit K_1.

For $u = K_2$, note that, considering the characteristic sufficiently close to the violation point (so that $u > 0$)

$$\begin{aligned} \frac{\partial u}{\partial \chi} = r_1(u^2 - u) + \theta q_1(v - u) &< \theta q_1(v - u) \\ &< \theta q_1(K_2 - u), \end{aligned}$$

so that u does not hit K_2.

Similarly for v, hence we are done. □

Finally, again using the inequality relaxation, and noting that if the initial data is identically zero then the solution is identically zero for all time, we have:

Lemma 3.6 *If $K_1 \leq u_0(x), v_0(x) \leq K_2$ for all x, for some constants $-\infty < K_1 \leq 0$ and $0 \leq K_2 \leq 1$, and (u_0, v_0) is in C^1, then C^1 solutions (u, v) of equations* (3.1), (3.2) *satisfy $K_1 \leq u, v \leq K_2$ for all time.*

No such result will be true for a pair of constants K_1 and K_2 both strictly on the same side of zero — the solution to the initial value problem with $u_0 = v_0$ identically equal to some constant K, such that $-\infty < K < 1$, tends monotonically to zero (see section 3.8). This fact is put together with comparison arguments in section 3.3 to give a much stronger result than Lemma 3.6. Lemma 3.6 is related to a scaling transformation detailed in section 9.4, which enables the McKean representation to be extended to cover negative initial data.

Given bounded initial data with an upper bound no greater than 1, we can read off the appropriate values of K_1 and K_2 by defining:

$$\begin{aligned} K_1 &= \min\Big(0, \inf_x(u_0(x)), \inf_x(v_0(x))\Big), \\ K_2 &= \max\Big(0, \sup_x(u_0(x)), \sup_x(v_0(x))\Big). \end{aligned}$$

Local existence of solutions with piecewise smooth initial data (i.e. continuous but with a finite number of jump discontinuities in the x-derivative) follows by approximating (in the supremum norm) piecewise smooth data by C^1 data — the limiting solutions are classical except on characteristics $x \pm t = x_0$ propagating from points x_0 of discontinuity of $\frac{\partial u_0}{\partial x}$, $\frac{\partial v_0}{\partial x}$ (see Courant and Hilbert [20, page 471] and Whitham [75, pages 127–130]).

3.3 Comparison results

Consider two solutions (u, v) and $(\tilde{u}, \tilde{v})$ of the equations (3.1),(3.2), with $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$, $\tilde{u}(0, x) = \tilde{u}_0(x)$, $\tilde{v}(0, x) = \tilde{v}_0(x)$. On the characteristics of the form $x + t =$ constant, we know that

$$\begin{aligned} \frac{\partial u}{\partial \chi} &= r_1(u^2 - u) + \theta q_1(v - u), \\ \frac{\partial \tilde{u}}{\partial \chi} &= r_1(\tilde{u}^2 - \tilde{u}) + \theta q_1(\tilde{v} - \tilde{u}), \end{aligned}$$

and similar equations for v and $\tilde{v}$ along the other characteristics.

Studying the difference between the solutions along a characteristic $x + t =$ constant, we see that it obeys the equation

$$\begin{aligned} \frac{\partial(u - \tilde{u})}{\partial \chi} &= r_1\big((u^2 - \tilde{u}^2) - (u - \tilde{u})\big) + \theta q_1\big((v - \tilde{v}) - (u - \tilde{u})\big), \\ &= (u - \tilde{u})\big(r_1(u + \tilde{u} - 1) - \theta q_1\big) + \theta q_1(v - \tilde{v}). \end{aligned}$$

Provided that the initial data (u_0, v_0) and $(\tilde{u}_0, \tilde{v}_0)$ satisfies the conditions of Lemma 3.6, then $(u + \tilde{u})$ will be bounded below (for all time), enabling us to write, along the characteristic up until a putative equality of u and $\tilde{u}$,

$$\frac{\partial(u - \tilde{u})}{\partial \chi} \geq C(u - \tilde{u}) + \theta q_1(v - \tilde{v}),$$

for some constant C. There is a similar equation for the difference of v and $\tilde{v}$.

Thus, if $u_0 > \tilde{u}_0$ and $v_0 > \tilde{v}_0$, then, for all time, $u > \tilde{u}$ and $v > \tilde{v}$. By taking sequences of initial data we obtain the following result.

Lemma 3.7 *If (u, v) and $(\tilde{u}, \tilde{v})$ are C^1 solutions of equations (3.1),(3.2) with $u_0 \geq \tilde{u}_0$ and $v_0 \geq \tilde{v}_0$, then, for all time, $u \geq \tilde{u}$ and $v \geq \tilde{v}$.*

Since the solution to the initial value problem with $u_0 = v_0$ identically equal to some constant K, such that $-\infty < K < 1$, tends monotonically to zero (see section 3.8), then any solution bounded between $-\infty < K_1 \leq 0$ and $0 \leq K_2 < 1$ will tend to zero. Thus Lemma 3.6 can be modified to the following result.

Lemma 3.8 *If $K_1 \leq u_0(x), v_0(x) \leq K_2$ for all x, for some constants $-\infty < K_1 \leq 0$ and $0 \leq K_2 < 1$, and (u_0, v_0) is in C^1, then C^1 solutions (u, v) of equations (3.1), (3.2) satisfy $K_1 \leq u, v \leq K_2$ for all time and $u(t, x)$ and $v(t, x) \to 0$ (uniformly in x) as $t \to 0$.*

3.4 Weak solutions and the Rankine-Hugoniot jump conditions

To deal with discontinuous initial data it is necessary to utilise the concept of *weak* solution (see Courant and Hilbert [20, pages 620–636], John [39, pages 141–145], Lax [45], Logan [47, Chapter 3] and Whitham [75, pages 39–42]). This is a solution which satisfies the system in an integrated sense (which implies that, if smooth, it is a classical solution).

DEFINITION. A *test function* is a 2-vector-valued function $\Phi := \big(\phi_1(t,x), \phi_2(t,x)\big)$ such that each of ϕ_1 and ϕ_2 is infinitely differentiable with compact support (i.e. $\phi_1, \phi_2 \in C_0^\infty$).

DEFINITION. For u, v bounded and measurable, we say that $U := (u, v)$ is a *weak solution* of equation (2.1) if, for all *test functions* Φ,

$$\int_{t=0}^{\infty}\int_{x=-\infty}^{\infty}\Big(U.\Phi_t - BU.\Phi_x + \big(R(U^2-U)+\theta QU\big).\Phi\Big)dtdx + \int_{x=-\infty}^{\infty} U(0,x).\Phi(0,x)dx = 0. \quad (3.3)$$

Thus, for a smooth classical solution (u, v), multiplying through the differential equation (2.1) by a test function and then integrating (by parts, moving the derivatives onto the test function) shows that it is also a weak solution. Conversely, a weak solution that is smooth is also a classical solution (i.e. these two notions of solution are equivalent for smooth (u, v)). Going from weak to classical is done by choosing a test function supported on $t > 0$, then integration by parts (which is permissible since u, v and Φ are smooth) gives

$$\int_{t=0}^{\infty}\int_{x=-\infty}^{\infty}\big(U_t - BU_x + R(U^2-U) + \theta QU\big).\Phi dtdx = 0$$

which implies that $U_t - BU_x + R(U^2 - U) + \theta QU = 0$ (using the density of the test functions).

However, the notion of a weak solution is more general than that of a classical solution. For example, a function which is a smooth classical solution on parts of the half-plane $t > 0, x \in \mathbb{R}$ but which also has curves of discontinuity is certainly not a classical solution, but may be a weak solution. These curves of discontinuity are referred to as the *shocks* in the system.

For such a piecewise classical solution to be a weak solution its curves of discontinuity in the xt-plane must satisfy certain conditions, known as the *jump*, *shock* or *Rankine-Hugoniot conditions.* They relate the values of the solution ahead and behind the discontinuity to the speed of the discontinuity itself. These conditions arise from integrating by parts with a test function supported on a region containing discontinuities in the solution.

The Rankine-Hugoniot jump conditions imply that a piecewise classical solution is a weak solution and tie up perfectly with the probabilistic representation of the solution (this and other features of the solution are discussed probabilistically in section 6.6). For a semi-linear system such as (2.1) they imply that a component of the weak solution can only have discontinuities across characteristics corresponding to that component (shock paths can only be characteristic curves). The derivation of these conditions for piecewise classical solutions to the system (2.1) is done in section 3.5.

Moving to the weak setting requires further care since jump conditions are not necessarily sufficient to guarantee uniqueness of solutions. Lemma 3.10 proves existence and uniqueness of *weak* solutions for a certain class of bounded, measurable initial data.

3.5 The jump conditions for the system (2.1)

We follow the derivations of Logan [47, pages 110–112] and Smoller [67, pages 247–248] using our system (2.1).

Firstly, recall Green's Theorem in the plane:

Theorem 3.9 *Let C be* [illegible] *curve in the tx-plane, and let D denote the domain enclos*$\int_C pdx + qdt = \int\int_D (q_x - p_t)dxdt$ *x) are smooth functions in $D \cup C$, then*

$$\int_C pdx + qdt = \int\int_D (q_x - p_t)dxdt$$

where the line integral over C is taken in the counterclockwise direction.

Now, let $U := (u, v)$ be a *weak* solution of (2.1), smooth except possibly across Γ, a smooth curve in spacetime given by $x = s(t)$, and continuous up to Γ from both sides. Let D be a region (suitable for the application of Green's Theorem) centred at some point on Γ and lying in the $t > 0$ plane, chosen sufficiently small so that is split into only two disjoint subsets, D_1 and D_2, by Γ, and Γ intersects the boundary of D exactly twice (at P_1 and P_2, defined so that the time-coordinate at P_1 is less than that at P_2).

Now, for any *test function* $\Phi = (\phi, \psi)$ whose support lies within D, by equation (3.3),

$$\begin{aligned} 0 &= \int\int_D \Big(U.\Phi_t - BU.\Phi_x + \big(R(U^2 - U) + \theta QU\big).\Phi\Big)dtdx \\ &= \int\int_{D_1} \Big(U.\Phi_t - BU.\Phi_x + \big(R(U^2 - U) + \theta QU\big).\Phi\Big)dtdx \\ &+ \int\int_{D_2} \Big(U.\Phi_t - BU.\Phi_x + \big(R(U^2 - U) + \theta QU\big).\Phi\Big)dtdx. \end{aligned} \tag{3.4}$$

Rewriting the integrals in (3.4) using the fact the U is smooth off Γ gives

$$\begin{aligned} \int\int_{D_i} \Big(U.\Phi_t - BU.\Phi_x + \big(R(U^2 - U) + \theta QU\big).\Phi\Big)dtdx &= \int\int_{D_i} \big((U.\Phi)_t - B(U.\Phi)_x\big)dtdx \\ &= \int_{\partial D_i} -U.\Phi dx - BU.\Phi dt, \end{aligned}$$

where we have used Green's theorem in the last step.

Since $\Phi = 0$ on ∂D, the line integral is only non-zero along Γ. By looking at a test function for which ψ is identically zero, and one for which ϕ is identically zero, we obtain the conditions

$$\int_{\partial D_1} u.\phi dx + b_1 u.\phi dt + \int_{\partial D_2} u.\phi dx + b_1 u.\phi dt = 0, \tag{3.5}$$

$$\int_{\partial D_1} v.\psi dx + b_2 v.\psi dt + \int_{\partial D_2} v.\psi dx + b_2 v.\psi dt = 0. \tag{3.6}$$

Since ϕ is arbitrary equation (3.5) implies that u can only have a discontinuity across Γ if Γ is a characteristic for u — equation (3.5) places no other restriction on u. Similarly equation (3.6) implies only that discontinuities in v occur across characteristics for v.

Thus we conclude that a piecewise classical solution is a weak solution if discontinuities in u and v only occur across their respective characteristics or, equivalently, that Γ must be a characteristic curve, and that only the corresponding component of the solution can be discontinuous across Γ.

3.6 Heaviside initial data

We now construct explicitly a piecewise classical solution for Heaviside initial data that satisfies the Rankine-Hugoniot jump conditions (hence it is a weak solution and will be shown to be unique by Lemma 3.10).

For the Heaviside initial data the jump conditions reduce to two requirements — that the discontinuity in u propagates along the characteristic for u that goes through zero, i.e. u jumps across $x = -t$ and is continuous elsewhere, and that the discontinuity in v propagates along the characteristic for v that goes through zero, i.e. v jumps across $x = t$ and is continuous elsewhere. We have defined the Heaviside function so as to be left-continuous and construct so that our solution inherits this property (we are free to choose the type of continuity arbitrarily for a weak solution since its value on a line is not picked up in the defining integration) — this matches the continuity which the probabilistic approach implies. The probabilistic interpretation of this solution is discussed in section 6.6.

Clearly the solution for the Heaviside data is identically zero for $x \leq -t$ and identically one for $x > +t$, which we can show using domains of dependence. In these regions the solution depends only on a section of constant initial data, where the constant (zero or one) is such that the solution is obviously constant in time.

We have stated that we are constructing a left-continuous solution — therefore on $x = +t$, $u = 1$ while v remains to be calculated.

We integrate along the $x = +t$ characteristic to find v there. Since $u = 1$,

$$\frac{\partial v(t, +t)}{\partial \chi} = r_2(v^2 - v) + \theta q_2(1 - v)$$

and $v(0,0) = 0$. Hence, when $r_2 \neq \theta q_2$,

$$v(t, +t) = \frac{\theta q_2 \Big(\exp\big((r_2 - \theta q_2)t\big) - 1\Big)}{r_2 \exp\big((r_2 - \theta q_2)t\big) - \theta q_2},$$

and v increases from 0 to $\min\left(\frac{\theta q_2}{r_2}, 1\right)$ as t goes from 0 to ∞. When $r_2 = \theta q_2$,

$$v(t,+t) = \frac{r_2 t}{r_2 t + 1},$$

and v increases from 0 to 1 as t goes from 0 to ∞.

We can also investigate the discontinuity in u along $x = -t$ similarly. v is continuous across this characteristic and zero on it, so is zero on $x = -t+$. Thus, integrating along the inside edge of the characteristic,

$$\frac{\partial u(t,-t+)}{\partial \chi} = r_1(u^2 - u) - \theta q_2 u$$

and $u(0,0+) = 1$. Hence,

$$u(t,-t+) = \frac{r_1 + \theta q_1}{r_1 + \theta q_1 \exp((r_1 + \theta q_1)t)},$$

and u decreases from 1 to 0 as t goes from 0 to ∞.

We now know the values of both u and v on the inside edge of the wedge $|x| < t$, between the discontinuities — and this data is continuous and between 0 and 1. Thus there is a unique solution to the problem with these as initial/boundary values and this solution is between 0 and 1. It can be found by following characteristics $x + t =$ constant from the $x = +t$ characteristic. This solution is piecewise classical with the only discontinuities being across the characteristics as required and so is indeed a weak solution.

See Chapters 8 and 9 for results of computational work on this initial-value problem.

3.7 Discontinuous initial data

We follow the method of Beale [4] in his work on the Broadwell model. Let ϕ be a test function with

$$\phi \in C_0^\infty, \quad \phi \geq 0, \quad \int_{-\infty}^{\infty} \phi(x)dx = 1,$$

and let $\phi_k(x) = k\phi(kx)$, $k \in \mathbb{N}$. For non-smooth initial data (u_0, v_0) we consider initial data formed by mollifying (u_0, v_0). We use the convolutions $u_0^{(k)} = u_0 * \phi_k$ and $v_0^{(k)} = v_0 * \phi_k$ and check that the corresponding solutions of the PDE converge as $k \to \infty$. Note that if u_0 and v_0 are between constants K_1 and K_2, then so are the mollified versions, $u_0^{(k)}$ and $v_0^{(k)}$.

In the following argument we consider differences between solutions for different mollifications of the same measurable initial data. We work with initial data that is bounded between constants K_1 and K_2, such that $-\infty < K_1 \leq 0$ and $0 \leq K_2 \leq 1$, i.e. $(u_0(x), v_0(x)) \in [K_1, K_2]^2$ for all x. For the argument to work these differences between solutions should be in L^1, which we show follows if the difference between two different mollifications of the initial data is in L^1. This is clearly true for the Heaviside data, and for initial data that is itself in L^1. Then, for example, it is true for bounded initial data that only differs from the Heaviside function by an L^1-function.

With this in mind define a *step function* s to be a function of the form

$$s(x) = \begin{cases} k_1 & \text{if } x > 0, \\ k_2 & \text{if } x \leq 0, \end{cases}$$

for some constants k_1 and k_2 such that $-\infty < k_1, k_2 \leq 1$. Then if both components of the initial data differ from step functions (with possibly different constants) by L^1-functions, the difference between mollifications will be L^1. This requirement can be thought of as requiring that the initial data has limits at ∞ and $-\infty$ that are approached rapidly enough.

We prove the following result (which is our version of Lemma 3.1 of Beale [4]).

Lemma 3.10 *Let $(u_0(x), v_0(x)) \in [K_1, K_2]^2$ for all x, such that each of $u_0(x)$ and $v_0(x)$ differs from a step function by an L^1-function. Define $u_0^{(k)} = u_0 * \phi_k$ and $v_0^{(k)} = v_0 * \phi_k$ as above and let $(u^{(k)}, v^{(k)})$ be the solution of (2.1) with $(u^{(k)}(0), v^{(k)}(0)) = (u_0^{(k)}, v_0^{(k)})$. Then the $u^{(k)}$ and $v^{(k)}$ are between K_1 and K_2 for all t, x and, for $1 \leq p < \infty$, for $\tau > 0$ arbitrary, $(u^{(k)} - u^{(1)}, v^{(k)} - v^{(1)})$ converges in $C(0, \tau; L^p)$ to $(u - u^{(1)}, v - v^{(1)})$ where (u, v) is a weak solution of (2.1) with $(u(0), v(0)) = (u_0, v_0)$. This weak solution is also bounded by K_1 and K_2 and is unique.*

Proof. By Lemma 3.6, $K_1 \leq u^{(k)}(t,x) \leq K_2$ and $K_1 \leq v^{(k)}(t,x) \leq K_2$. Now, if $U = (u^{(k)} - u^{(k')}, v^{(k)} - v^{(k')})$, then, for $i = 1, 2$ and integers k, k',

$$U_{i,t} = (-1)^{i-1} U_{i,x} + h_i(t,x), \qquad |h_i(t,x)| \leq C|U(t,x)|, \tag{3.7}$$

where C depends on the parameters θ, q_1, q_2, r_1 and r_2, but not on k, k'. Define f_n by $f_n(y) = |y| - 1/(2n)$ for $|y| \geq 1/n$, $f_n(y) = ny^2/2$ for $|y| \leq 1/n$. Note that, for each n, f_n is continuous with continuous first derivative. Also $0 \leq f_n(y) \leq |y|$, and $f_n(y) \to |y|$ as $n \to \infty$.

Multiplying equation (3.7) by $f_n'(U_i)$, yields:

$$\big(f_n(U_i)\big)_t = (-1)^{i-1}\big(f_n(U_i)\big)_x + f_n'(U_i)h_i(t,x).$$

Integrating in x over a finite interval $[x_1, x_2]$ gives, for $i \in \{1, 2\}$,

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int_{x_1}^{x_2} f_n(U_i)dx\right) &\leq (-1)^{i-1}\left(f_n(U_i(t,x_2)) - f_n(U_i(t,x_1))\right) \\ &+ C\int_{x_1}^{x_2} |U_i(t,x)|dx. \end{aligned}$$

We now integrate on $[0,t]$ to get

$$\begin{aligned} \int_{x_1}^{x_2} f_n(U_i(t,x))dx &\leq \int_{x_1}^{x_2} f_n(U_i(0,x))dx + C\int_0^t \int_{x_1}^{x_2} |U_i(s,x)|dxds \\ &+ (-1)^{i-1}\int_0^t \Big(f_n(U_i(s,x_2)) - f_n(U_i(s,x_1))\Big)ds. \end{aligned}$$

Letting $n \to \infty$ and taking limits using the bounded convergence theorem implies that

$$\begin{aligned}\int_{x_1}^{x_2} |U_i(t,x)|dx &\leq \int_{x_1}^{x_2} |U_i(0,x)|dx + C\int_0^t \int_{x_1}^{x_2} |U_i(s,x)|dxds \\ &+ (-1)^{i-1} \int_0^t (|U_i(s,x_2)| - |U_i(s,x_1)|)ds \\ &\leq \int_{x=-\infty}^{\infty} |u_{i,0}^{(k)} - u_{i,0}^{(k')}|dx + C\int_0^t \int_{x_1}^{x_2} |U_i(s,x)|dxds \\ &+ 2t||U_i||_{L^\infty[\mathbb{R}\times[0,t]]},\end{aligned} \tag{3.8}$$

where in the latter step we are using the fact that the initial data is only an L^1-function from a step function to say that $\int_{x=-\infty}^{\infty} |u_0^{(k)} - u_0^{(k')}|dx$ is finite.

Hence, for any T,

$$\begin{aligned}\sup_{t\in[0,T]} \int_{x_1}^{x_2} |U_i(t,x)|dx &\leq \int_{x=-\infty}^{\infty} |u_0^{(k)} - u_0^{(k')}|dx + CT\left(\sup_{t\in[0,T]} \int_{x_1}^{x_2} |U_i(t,x)|dx\right) \\ &+ 2T||U_i||_{L^\infty[\mathbb{R}\times[0,T]]}.\end{aligned}$$

Choosing T sufficiently small so that $CT < 1$ and rearranging gives,

$$(1-CT)\left(\sup_{t\in[0,T]} \int_{x_1}^{x_2} |U_i(t,x)|dx\right) \leq \int_{x=-\infty}^{\infty} |u_0^{(k)} - u_0^{(k')}|dx + 2T||U_i||_{L^\infty[\mathbb{R}\times[0,T]]},$$

where the constant on the right-hand side (which we will now denote by γ) does not depend on x_1 and x_2 (γ is finite due to boundedness of solutions for smooth, bounded initial data — which the mollified initial data are). Thus, for $0 \leq t < T$,

$$\int_{-\infty}^{\infty} |U_i(t,x)|dx \leq \frac{\gamma}{1-CT} < \infty,$$

and hence each $U_i(t,\cdot) \in L^1((0,T)\times\mathbb{R})$. Now it is possible to return to equation (3.8) and improve our estimate. Firstly, rewrite it as:

$$\begin{aligned}\int_{x_1}^{x_2} |U_i(t,x)|dx &\leq \int_{x_1}^{x_2} |U_i(0,x)|dx + C\int_0^t \int_{x_1}^{x_2} |U_i(s,x)|dxds \\ &+ \int_0^t |U_i(s,x_2)|ds + \int_0^t |U_i(s,x_1)|ds.\end{aligned} \tag{3.9}$$

Since each $U_i(t,\cdot) \in L^1((0,T)\times\mathbb{R})$ there exist sequences such that, letting $x_1 \to -\infty$ and $x_2 \to \infty$ along these sequences, keeping t fixed, the latter two terms of equation (3.9) tend to zero. The fact that each $U_i(t,\cdot) \in L^1((0,T)\times\mathbb{R})$ keeps the other terms finite as $x_1 \to -\infty$ and $x_2 \to \infty$.

Hence we arrive at:

$$\int_{x=-\infty}^{\infty} |U(t,x)|dx \le \int_{x=-\infty}^{\infty} |u_0^{(k)} - u_0^{(k')}|dx + C\int_0^t \int_{x=-\infty}^{\infty} |U(s,x)|dxds.$$

Thus, from Gronwall's inequality, $|U(t,\cdot)|_{L^1} \le e^{Ct}|u_0^{(k)} - u_0^{(k')}|_{L^1}$. Since $(u_0^{(k)} - u_0^{(1)}, v_0^{(k)} - v_0^{(1)}) \to (u_0 - u_0^{(1)}, v_0 - v_0^{(1)})$ in $L^1(\mathbb{R})$, it follows that $(u^{(k)} - u^{(1)}, v^{(k)} - v^{(1)})$ converges in $C\big(0,\tau; L^1(\mathbb{R})\big)$ to a function $w(t,x) = \big(w_1(t,x), w_2(t,x)\big)$. In fact, since the $u^{(k)}$ and $v^{(k)}$ are all bounded between K_1 and K_2, the convergence takes place in $C\big(0,\tau; L^p(\mathbb{R})\big), 1 \le p < \infty$. Then it is easily seen that $\big(u(t,x), v(t,x)\big) := w(t,x) + \big(u^{(1)}(t,x), v^{(1)}(t,x)\big)$ is a weak solution of the equation and that (u,v) is again bounded by K_1 and K_2 for all time.

We now verify the uniqueness property. For these solutions we can show, for u that, for arbitrary $\phi \in C_0^\infty(\mathbb{R})$ and $t > 0$,

$$\int_{-\infty}^{\infty} u(t,x+t)\phi(x)dx - \int_{-\infty}^{\infty} u_0(x)\phi(x)dx =$$

$$\int_{-\infty}^{\infty}\int_0^t \Big(r_1\big(u(s,x+s)^2 - u(s,x+s)\big) + \theta q_1\big(v(s,x+s) - u(s,x+s)\big)\Big)\phi(x)dsdx$$

and similarly for v.

Therefore

$$u(t,x+t) - u_0(x) =$$

$$\int_0^t \Big(r_1\big(u(s,x+s)^2 - u(s,x+s)\big) + \theta q_1\big(v(s,x+s) - u(s,x+s)\big)\Big)ds$$

for almost all (t,x). Hence if there are two solutions with the same initial data and $y(t)$ is the L^∞-norm of the difference at time t, we obtain (using the boundedness between K_1 and K_2) an estimate

$$y(t) \le c\int_0^t y(s)ds, \qquad y(0) = 0,$$

and this implies y is identically zero. □

The restriction to initial data that is an L^1 function different from a step function can be removed using domains of dependence and a truncation argument as follows.

The value of u on the line (T,X) to $(T,X+1)$ depends only on the initial data in the interval $[X-T, X+T+1]$. Thus it should agree with the solution whose initial data is identically zero outside this interval, and matches on the interval, which we shall describe as the *truncated* initial data. However the truncated initial data satisfies the conditions of Lemma 3.10 and so the truncated initial data has a unique corresponding solution. Thus we can *define* the solution for more general initial data to be that constructed by piecing together solutions for the truncated initial data, Lemma 3.10 guarantees that this will be well-defined.

3.8 The case when $b_1 = b_2$

When $b_1 = b_2$ we change to moving coordinates at speed b_1. Relabelling as before we obtain the ODE:

$$\begin{aligned} u_t = &\ r_1(u^2 - u) + \theta q_1(v - u) = &\ f(u, v); \\ v_t = &\ r_2(v^2 - v) + \theta q_2(u - v) = &\ g(u, v). \end{aligned} \tag{3.10}$$

For this pair of equations the square $[0, 1]^2$ is positively invariant — since $f(0, v) > 0$ and $f(1, v) < 0$ for $0 < v < 1$ and similarly $g(u, 0) > 0$ and $g(u, 1) < 0$ for $0 < u < 1$. S and T are clearly equilibria. At all other boundary points the flow is into the unit square in forwards time. In fact, any square of the form $[K_1, K_2]^2$ for $-\infty < K_1 \leq 0$ and $0 \leq K_2 \leq 1$ is positively invariant.

For the Heaviside initial data the solution is clear — the Heaviside function simply propagates at speed $-b_1$. This is also clear from the probabilistic method (see section 6.6).

The pair of equations (3.10) is also relevant because, if the initial data in the PDE system (equation (2.1)) is constant, that is $u(0, x) = K_1, v(0, x) = K_2$ (independently of x), then the solution satisfies (3.10), with initial conditions $u(0) = K_1, v(0) = K_2$. The constant initial data case is particularly relevant in light of the comparison arguments in section 3.3. Solutions with initial data bounded below and above by constants K_1 and K_2 can be bounded below and above by the solutions for constant initial data K_1 and K_2, but these solutions clearly go to zero for $-\infty < K_1 \leq 0$ and $0 \leq K_2 < 1$ (since the square $[K_1, K_2]^2$ is positively invariant and contains only one equilibrium, which is S).

Chapter 4

Useful algebraic results

Lemma 4.1 *For $\theta > 0$, there exists a (finite) number $c(\theta)$ that satisfies* $\min(-b_1, -b_2) \leq c(\theta) \leq \max(-b_1, -b_2)$ *and such that*

(i) *for $c > c(\theta)$, $c \neq \max(-b_1, -b_2)$, the matrix $K_{c,\theta}(T)$ has at least one stable monotone eigenvalue, and*

(ii) *for $c < c(\theta)$, $c \neq \min(-b_1, -b_2)$, the matrix $K_{c,\theta}(T)$ has no stable monotone eigenvalues.*

Proof. In fact, we can say rather more than this about $K_{c,\theta}(T)$. Explicitly, if $(B + cI)$ is invertible,

$$K_{c,\theta}(T) = \begin{pmatrix} \frac{\theta q_1 - r_1}{b_1 + c} & -\frac{\theta q_1}{b_1 + c} \\ -\frac{\theta q_2}{b_2 + c} & \frac{\theta q_2 - r_2}{b_2 + c} \end{pmatrix}.$$

Thus, its two eigenvalues, λ^+ and λ^-, are given by:

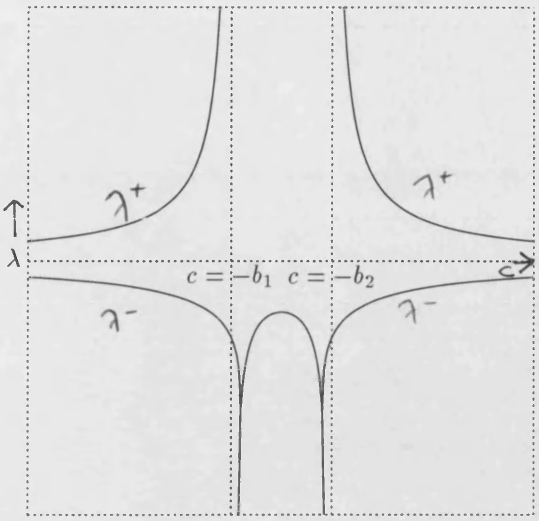
$$\lambda^{\pm} = \frac{1}{2}\left\{ \frac{\theta q_1 - r_1}{b_1 + c} + \frac{\theta q_2 - r_2}{b_2 + c} \pm \sqrt{\left(\frac{\theta q_1 - r_1}{b_1 + c} - \frac{\theta q_2 - r_2}{b_2 + c}\right)^2 + \frac{4\theta^2 q_1 q_2}{(b_1 + c)(b_2 + c)}} \right\}. \tag{4.1}$$

These eigenvalues correspond to eigenvectors of the form $\begin{pmatrix} 1 \\ v^{\pm} \end{pmatrix}$, where

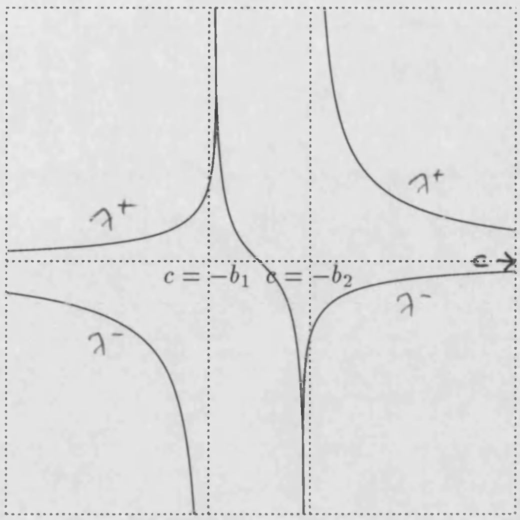
$$v^{\pm} = \frac{(\theta q_1 - r_1) - \lambda^{\pm}(b_1 + c)}{\theta q_1}. \tag{4.2}$$

Bifurcation diagrams of the eigenvalues as functions of c are given in Figures 4-1 and 4-2.

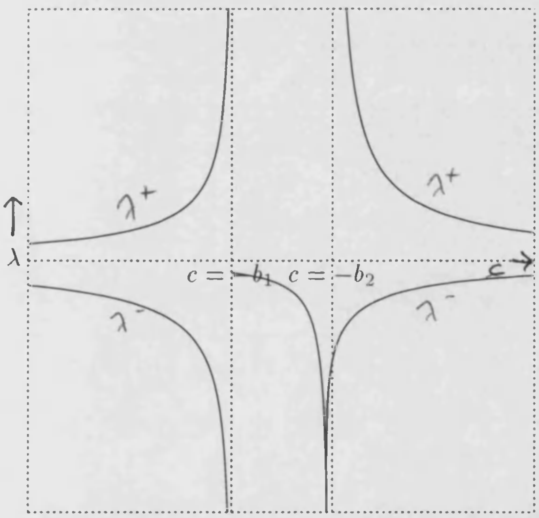
For $c > \max(-b_1, -b_2)$, note that λ^- is a stable monotone eigenvalue and λ^+ is not, since $v^+ < 0$. For $c < \min(-b_1, -b_2)$, note that neither eigenvalue is stable monotone as $v^- < 0$ and $\lambda^+ > 0$. Thus, if there is a critical value $c(\theta)$ it lies in the interval claimed. For $b_1 = b_2 = b$, say, this is enough to prove the lemma, $c(\theta) = -b$, independent of θ.



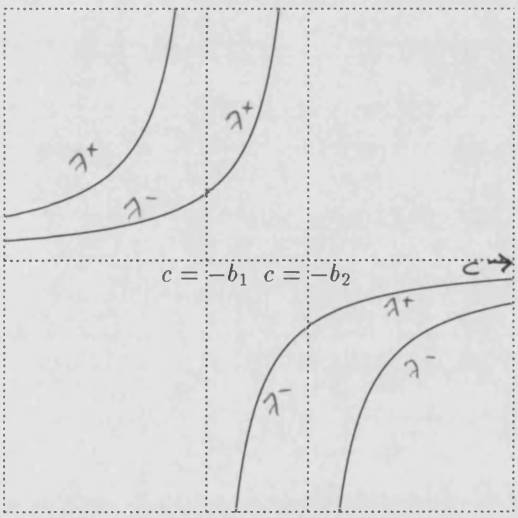
(a) $\theta q_1 = 1, \theta q_2 = 2, r_1 = 2, r_2 = 1.$



(b) $\theta q_1 = 2, \theta q_2 = 2, r_1 = 1, r_2 = 1.$

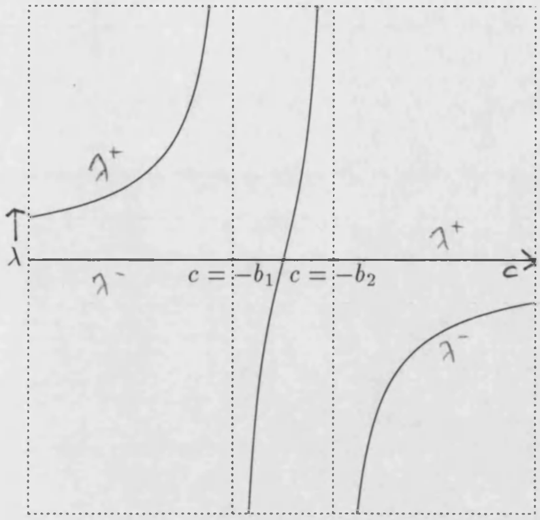


(c) $\theta q_1 = 2, \theta q_2 = 2, r_1 = 2, r_2 = 1.$

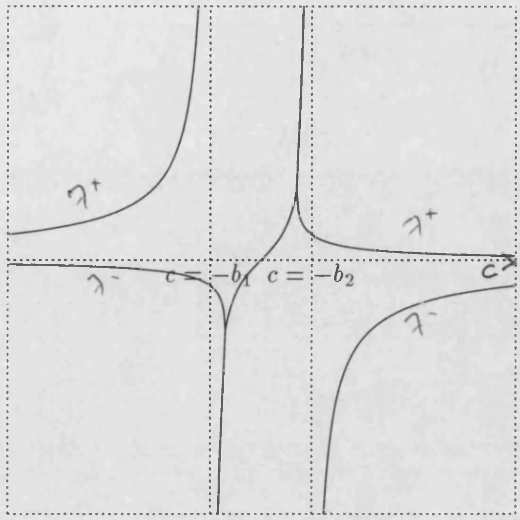


(d) $\theta q_1 = 1, \theta q_2 = 1, r_1 = 4, r_2 = 4.$

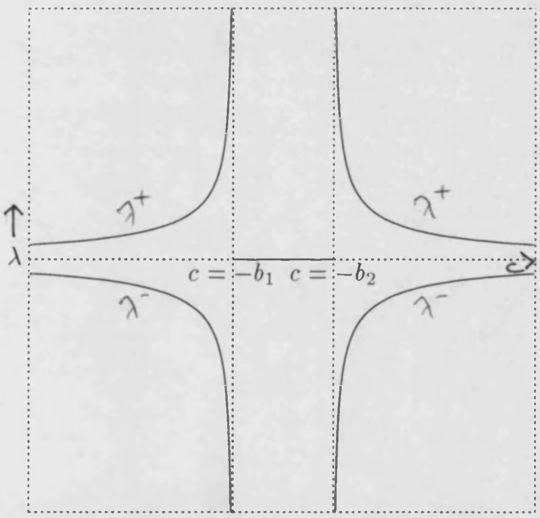
Figure 4-1: Bifurcation diagrams — plots of the eigenvalues λ^+ and λ^-, of $K_{c,\theta}(T)$, against c (where the eigenvalues are a complex conjugate pair the real part is plotted). The asymptotes $c = -b_1$ and $c = -b_2$ are marked. All diagrams assume that $b_1 > b_2$. $c(\theta) = -b_2$ in (d), and is slightly smaller than that in each of (a)-(c) — it is the right-hand intersection of λ^+ and λ^-.



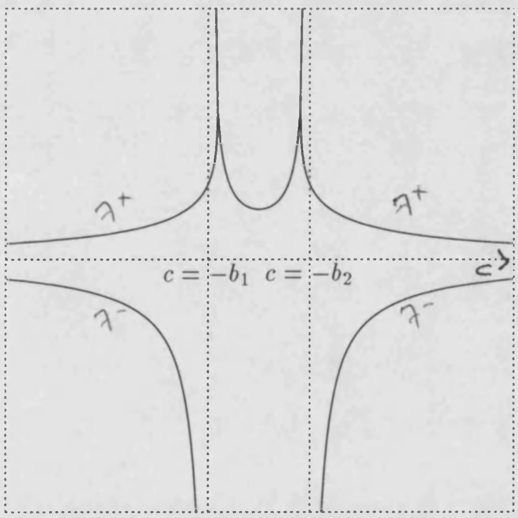
(a) $\theta q_1 = 2, \theta q_2 = 2, r_1 = 4, r_2 = 4.$



(b) $\theta q_1 = 1, \theta q_2 = 2, r_1 = 2, r_2 = 3.$



(c) $\theta q_1 = 2, \theta q_2 = 1, r_1 = 2, r_2 = 1.$



(d) $\theta q_1 = 2, \theta q_2 = 1, r_1 = 1, r_2 = 2.$

Figure 4-2: Bifurcation diagrams — plots of the eigenvalues λ^+ and λ^-, of $K_{c,\theta}(T)$, against c (where the eigenvalues are a complex conjugate pair the real part is plotted). The asymptotes $c = -b_1$ and $c = -b_2$ are marked. All diagrams assume that $b_1 > b_2$. $c(\theta) = -b_2$ in each of these plots.

So, to complete the proof of the lemma it is sufficient to consider the case $b_1 > b_2$, where it is necessary to examine the eigenvalues when $-b_1 < c < -b_2$, i.e. $(b_1 + c) > 0 > (b_2 + c)$.

Examining the term under the square root in equation (4.1) (which we will denote by $h(c)$, for fixed θ), for $-b_1 < c < -b_2$, note that it can have one, two or no zeroes. If there are two zeroes let us denote them by c_1 and c_2, with $-b_1 < c_1 < c_2 < -b_2$. When $\theta q_2 > r_2$ and $\theta q_1 \neq r_1$, there are two zeroes of h and for $c_1 < c < c_2$, $h(c) < 0$. So in this case the eigenvalues of $K_{c,\theta}(T)$ are complex. When $-b_1 < c < c_1$ and when $c_2 < c < -b_2$, $h(c) > 0$, so the eigenvalues are real. When $c_2 < c < -b_2$ both eigenvalues are stable monotone. When $-b_1 < c < c_1$ the eigenvalues are not stable monotone — $\theta q_1 < r_1$ implies that the eigenvectors have components of opposite signs (as shown in Figure 4-1 (a)) while $\theta q_1 > r_1$ implies that the eigenvalues have positive real parts (as shown in Figure 4-1 (b)). Thus, in this case, $c(\theta) = c_2$.

When $\theta q_2 > r_2$ and $\theta q_1 = r_1$ there is a single zero of h, at c_1 say (as shown in Figure 4-1 (c)). When $c_1 < c < -b_2$, $h(c) > 0$ and both eigenvalues are stable monotone. When $-b_1 < c < c_1$, $h(c) < 0$ and so the eigenvalues are complex. Hence $c(\theta) = c_1$ in this case.

When $\theta q_2 \leq r_2$ matters are simpler because $\theta q_1 < r_1$ implies that the eigenvectors have components of opposite signs or the eigenvalues are complex, within the interval $-b_1 < c < -b_2$ (as shown in Figures 4-1 (d) and 4-2 (a) and (b)), and $\theta q_1 \geq r_1$ implies that the eigenvalues have non-negative real part throughout the interval $-b_1 < c < -b_2$ (as shown in Figures 4-2 (c) and (d)). Thus, for $\theta q_2 \leq r_2$, $c(\theta) = -b_2$.

Hence the lemma is proven. □

When $c = c(\theta)$ there is a repeated stable monotone eigenvalue, unless $c(\theta) = -b_i$, in which case the point is moot since the stability matrix does not exist, but the analysis can be completed by direct methods (see section 5.6).

For the probabilistic method we need to look at one of these special cases in more detail — the case where $-b_i < c(\theta) < -b_j$ and $c = -b_j$ (where $\{i, j\} = \{1, 2\}$). Assume, without loss of generality, that $b_1 > b_2$. Then $-b_1 < c(\theta) < -b_2$ if and only if $r_2 < \theta q_2$.

Recall that $K_{c,\theta}(T)$ is defined by equation (2.4), thus an eigenvector v (with corresponding eigenvalue λ) of $K_{c,\theta}(T)$ will satisfy the equation

$$\lambda(B + cI)v + (R + \theta Q)v = 0. \tag{4.3}$$

For $(B + cI)$ invertible, this relation is also true in the opposite direction — a non-trivial vector v that satisfies equation (4.3) is an eigenvector of $K_{c,\theta}(T)$ with eigenvalue λ. However, for $(B + cI)$ singular, equation (4.3) can still have non-trivial solutions, in the case of interest it has one solution,

$$\lambda = -\frac{(r_1 - \theta q_1)(r_2 - \theta q_2) - \theta^2 q_1 q_2}{(b_1 - b_2)(r_2 - \theta q_2)} < 0,$$

with $v = \begin{pmatrix} 1 \\ v_2 \end{pmatrix}$, where

$$v_2 = \frac{\theta q_2}{\theta q_2 - r_2} > 0.$$

Thus this solution (which is what we are really interested in, analysis of $K_{c,\theta}(T)$ is a short-cut, and simplifies discussion by giving us a way of referring to eigenvalues and eigenvectors as those of $K_{c,\theta}(T)$) can be thought of as a generalized stable monotone eigenvalue of $K_{c,\theta}(T)$ and we have proved that, for $c > c(\theta)$ the matrix $K_{c,\theta}(T)$ has at least one (possibly generalized) stable monotone eigenvalue.

The value of $c(\theta)$ can also be derived from the probabilistic model of the system: the theory of large deviations gives a formula for $c(\theta)$ which effectively summarises all these cases; see section 6.3. That formula, which is shown in section 6.3 to be entirely equivalent to the preceding characterization, makes it easy to observe that $c(\theta)$ is decreasing as θ increases and that

$$c^* := \lim_{\theta \to \infty} c(\theta) = -\left(\frac{b_1 q_2 + b_2 q_1}{q_1 + q_2}\right).$$

This limit will be discussed in section 6.3, for now note that the lower bound on $c(\theta)$ given by Lemma 4.1 can therefore be tightened to c^*.

It is also possible to arrive at this limit by further manipulating $h(c)$. Rearranging the equation $h(c) = 0$ to express it as a quadratic in c (where the coefficients are quadratic in the other parameters, including θ) enables us to consider the limit as $\theta \to \infty$ by picking out the terms of highest order (i.e. order two in θ) only. Simplifying this expression yields

$$\theta^2 \left\{ c(\theta)(q_1 + q_2) + (b_1 q_2 + b_2 q_1) \right\}^2 + \text{terms of lower order in } \theta = 0.$$

Thus the θ^2 term is zero if, and only if, $c = c^*$.

Further explicit calculation on $K_{c,\theta}(S)$ and $K_{c,\theta}(T)$ allows us to summarise the locations (i.e. left- or right-half plane) of the eigenvalues as follows:

- For $c > \max(-b_1, -b_2)$: both eigenvalues of $K_{c,\theta}(S)$ are real and positive, the smaller is unstable monotone, the larger not; $K_{c,\theta}(T)$'s eigenvalues are both real, for $\theta < \frac{1}{\rho_1+\rho_2}$ both are negative, for $\theta = \frac{1}{\rho_1+\rho_2}$ one is zero and the other is negative and for $\theta > \frac{1}{\rho_1+\rho_2}$ one is positive and one is negative.

- For $c < \min(-b_1, -b_2)$: both eigenvalues of $K_{c,\theta}(S)$ are real and negative, the eigenvalue closer to zero is stable monotone, the other not; $K_{c,\theta}(T)$'s eigenvalues are real and positive for $\theta < \frac{1}{\rho_1+\rho_2}$, for $\theta = \frac{1}{\rho_1+\rho_2}$ one is positive and the other is zero and for $\theta > \frac{1}{\rho_1+\rho_2}$ one is positive and one is negative.

- For $\min(-b_1, -b_2) < c < \max(-b_1, -b_2)$: the eigenvalues of $K_{c,\theta}(S)$ are real and have opposite signs, one is stable monotone, the other unstable monotone; for $\theta < \frac{1}{\rho_1+\rho_2}$, $K_{c,\theta}(T)$'s eigenvalues are real and have opposite signs, details for larger θ depend on the relative sizes of c and $c(\theta)$.

The following result is required in our proof of $\mathcal{L}^1$ convergence of Z_λ in Theorem 6.4.

Lemma 4.2 *(i) Suppose that $c > c(\theta)$. Let $\lambda_s(c)$ be the stable monotone eigenvalue of $K_{c,\theta}(T)$ (the one nearer to 0 if there are two). From the definition of $K_{c,\theta}$ and $\lambda_s(c)$ it is easily seen* *that $-\lambda_s(c)c$ is the Perron-Frobenius eigenvalue of $(\lambda_s(c)B + \theta Q + R)$. For $\mu < \lambda_s(c)$, with μ sufficiently close to $\lambda_s(c)$,*

$$\Lambda_{PF}(\mu B + \theta Q + R) = -\mu c_1(\mu) \text{ for some } c_1(\mu) < c.$$

(ii) As $c \downarrow c(\theta)$, we have

$$\lambda_s(c)^{-1}\Lambda_{PF}(\lambda_s(c)B + \theta Q + R) \to -c(\theta) \text{ and } \left[\frac{\partial}{\partial \mu}\Lambda_{PF}(\mu B + \theta Q + R)\right]_{\mu=\lambda_s(c)} \to -c(\theta).$$

(iii) When $c(\theta) < \max(-b_1, -b_2)$, $K_{c(\theta),\theta}(T)$ has a double eigenvalue, which we will denote by λ_0. This eigenvalue is geometrically simple, i.e. it has only one normalised eigenvector even though it has algebraic multiplicity two, and is stable monotone.

Proof. We follow the proof of Lemma 4.4 of Crooks [21].

(i) We can explicitly write $\lambda_s(\cdot)$ as a function of c — for $c > \max(-b_1, -b_2)$ it is λ^- from equation (4.1), for $c < \max(-b_1, -b_2)$ it is λ^+ from equation (4.1). Considered as a function of c it is easy to see that it is continuously differentiable away from $c = -b_i$ for $i = 1, 2$ and to check that $\lambda_s(c) \downarrow \lambda_0$ as $c \downarrow c(\theta)$. So for $c > c(\theta)$ and $\lambda_0 < \mu < \lambda_s(c)$, there exists some $c_1(\mu)$ such that $c(\theta) < c_1(\mu) < c$ and $\lambda_s(c_1(\mu)) = \mu$ and thus $\Lambda_{PF}(\mu B + \theta Q + R) = -\mu c_1(\mu)$.

(ii) The first part follows from the fact that $\lambda_s(c)^{-1}\Lambda_{PF}(\lambda_s(c)B + \theta Q + R) = c$. For the second part consider again the explicit form of the Perron-Frobenius eigenvalue as a function of μ:

$$\begin{aligned}\Lambda_{PF}(\mu B + \theta Q + R) &= \frac{1}{2}(\mu(b_1 + b_2) - (\theta q_1 - r_1) - (\theta q_2 - r_2)) \\ &+ \frac{1}{2}\sqrt{\Big(\mu(b_1 - b_2) - (\theta q_1 - r_1) + (\theta q_2 - r_2)\Big)^2 + 4\theta^2 q_1 q_2}.\end{aligned}$$

It is clear, by continuity (considering Λ_{PF} as a function of μ and noting that the term under the square root is strictly positive), that

$$\left[\frac{\partial}{\partial \mu}\Lambda_{PF}(\mu B + \theta Q + R)\right]_{\mu=\lambda_s(c)} \to \left[\frac{\partial}{\partial \mu}\Lambda_{PF}(\mu B + \theta Q + R)\right]_{\mu=\lambda_0}$$

as $c \downarrow c(\theta)$ because $\lambda_s(c) \to \lambda_0$ as $c \downarrow c(\theta)$. Now

$$\frac{\partial}{\partial \mu}\Lambda_{PF}(\mu B + \theta Q + R) = \frac{\partial}{\partial \mu}\Lambda_{PF}(\mu B + \mu c(\theta) + \theta Q + R) - c(\theta). \tag{4.4}$$

We claim that $\Lambda_{PF}(\mu B + \mu c(\theta)I + \theta Q + R)$ must attain a local minimum at $\mu = \lambda_0$. Note that, for fixed $c > c(\theta)$, sufficiently close to $c(\theta)$, there are two values of μ such that $\Lambda_{PF}(\mu B + \mu c I + \theta Q + R) = 0$, these two values of μ are the two stable monotone eigenvalues of $K_{c,\theta}(T)$. We now use a convexity result due to Cohen [19].

To be precise, Cohen's result states that the Perron-Frobenius eigenvalue of a matrix $M_1 + M_2$ is a convex function of M_2, where M_1 has positive elements off the main diagonal and M_2 is a diagonal matrix (possibly 0). (Crooks [21] gives references for more elementary proofs than Cohen's original one.) Thus, for μ between the two eigenvalues $\Lambda_{PF}(\mu B + \mu c I + \theta Q + R) < 0$ (strict inequality since there can be at most two zeroes — each is an eigenvalue of $K_{c,\theta}(T)$, a 2×2 matrix). Thus, for $c = c(\theta)$, $\Lambda_{PF}(\mu B + \mu c(\theta) I + \theta Q + R) > 0$ except at $\mu = \lambda_0$ where it is zero.

Thus the derivative on the right hand side of equation (4.4) is zero at $\mu = \lambda_0$ and hence the proof of (ii) is complete.

(iii) As c decreases through $c(\theta)$ the two stable monotone eigenvalues of $K_{c,\theta}$ coalesce and, at least for c sufficiently close to $c(\theta)$, become a complex conjugate pair with negative real part. From equation (4.1), $\lambda_0 = \frac{1}{2}\left\{\frac{\theta q_1 - r_1}{b_1 + c(\theta)} + \frac{\theta q_2 - r_2}{b_2 + c(\theta)}\right\}$, and from equation (4.2), this corresponds to an eigenvector of the form $\begin{pmatrix} 1 \\ v_0 \end{pmatrix}$, where

$$v_0 = \frac{(\theta q_1 - r_1) - \lambda_0 (b_1 + c(\theta))}{\theta q_1}.$$

We can verify directly that λ_0 and v_0 as given above have the correct signs (negative and positive respectively) by using inequalities arising from the definition of $c(\theta)$. □

For the probabilistic proof of uniqueness, modulo translation, of monotone travelling waves from S to T, the next lemma is important.

Lemma 4.3 *Suppose that $c > c(\theta)$ and that $K_{c,\theta}(T)$ has two stable monotone eigenvalues. Let β be the stable monotone eigenvalue further from 0. Then, for $\alpha > \beta$ with α sufficiently close to β, the only non-negative 2-vector g such that*

$$0 \leq \big(\alpha(B + cI) + \theta Q + R\big) g$$

is the zero vector: $g = 0$.

Proof. Convexity of $\Lambda_{PF}(\mu(B + cI) + \theta Q + R)$ as a function of μ given by Cohen's Theorem [19], for $\beta < \mu < \lambda$, yields $\Lambda_{PF}(\mu(B + cI) + \theta Q + R) \leq 0$. However, there can only be two values (μ_1 and μ_2, say) of μ at which this function is 0 since each μ_i is thus an eigenvalue of a 2×2 matrix. These two values are therefore β and λ and the inequality is strict.

Suppose that for some μ satisfying $\beta < \mu < \lambda$, there exists g, non-negative, with $g \neq 0$, and $(\mu(B + cI) + \theta Q + R) g > 0$. Then $\Lambda_{PF}(\mu(B + cI) + \theta Q + R) \geq 0$ which is a contradiction. □

Finally, we need the following result in showing that

$$\mathbb{P}_{x,y}\big[Z_\lambda(\infty) = 0\big] = 0 \text{ or } 1.$$

Lemma 4.4 *If w is a 2-vector such that $0 \leq w \leq 1$ and*

$$R(w^2) = (R - \theta Q)w,$$

then either $w = (1,1)$ or $w = (0,0)$.

Proof. Rearranging the above equation notice that the problem amounts to searching for intersections of two parabolae inside the unit square — $\theta q_1 w_2 = (r_1 + \theta q_1)w_1 - r_1 w_1^2$ and $\theta q_2 w_1 = (r_2 + \theta q_2)w_2 - r_2 w_2^2$. Since each parabola goes through $w = (0,0)$ and $w = (1,1)$ there can be no other intersections in the square — since from $(0,0)$ to $(1,1)$ the curve $\theta q_1 w_2 = (r_1 + \theta q_1)w_1 - r_1 w_1^2$ is above the line $w_2 = w_1$ while the other curve is below it. □

Chapter 5

Analytic proofs of existence and uniqueness of travelling waves

5.1 Plan of attack

To prove Theorem 2.1 we use shooting arguments. First note that the path of any solution w of (2.2) which satisfies

$$w(x) \to T \quad \text{as} \quad x \to \infty, \quad w(x) \to S \quad \text{as} \quad x \to -\infty \quad \text{and} \quad w'(x) > 0, x \in \mathbb{R},$$

must lie entirely inside the open unit square. Whether it must also lie between the nullclines $w_1' = 0$ and $w_2' = 0$ depends on the values of various parameters, as explained below. Depending on the geometric configuration of the nullclines, we introduce shooting boxes. We observe that these regions have four key properties upon which our proof will be based:

- Non-constant solution curves do not intersect the boundary of the region tangentially and hence any non-constant solution curve which intersects the boundary crosses it transversally;
- Exit-times of solutions in the regions are continuous functions of initial conditions;
- No non-constant solution curve passes through S or T;
- Other than S and T there are no equilibria in the closure of these regions.

There are two types of region that we use — *inside regions* when $c > \max(-b_1, -b_2)$ and *outside regions* when $\min(-b_1, -b_2) < c < \max(-b_1, -b_2)$. The terminology is based on the geometry of the phase plane as will be seen shortly.

5.2 Labelling the phase plane

In the 2-dimensional $w = (w_1, w_2)$-plane consider the parabolae

$$P_1 : \theta q_1 w_2 - (r_1 + \theta q_1) w_1 + r_1 w_1^2 = 0$$

and

$$P_2 : \theta q_2 w_1 - (r_2 + \theta q_2) w_2 + r_2 w_2^2 = 0$$

and let Ω_i denote the open region in the first quadrant between P_i and the w_i-axis, $i = 1, 2$. The parabolae are of course the nullclines for the system, and their intersections are its equilibria. The point $S = (0, 0) \in P_1 \cap P_2$ for all $\theta > 0$, the point $T = (1, 1) \in P_1 \cap P_2$ for all $\theta > 0$ and $E_\pm \in P_1 \cap P_2$ if $0 < \theta \leq (4\rho_1\rho_2)^{-\frac{1}{2}}$.

Note that the relative positions of E_+, E_- and T depend on the value of θ and that no two of them are commensurate with respect to the partial ordering on $\mathbb{R}^2$ induced by the positive quadrant. Let us denote by E_1 the element of $\{E_+, E_-, T\}$ with the largest w_1-component and by E_3 that with the largest w_2-component. Then, for $\theta(\rho_1 + \rho_2) < 1$, $T = E_2$. When $\theta(\rho_1 + \rho_2) > 1$ and $\theta \leq (4\rho_1\rho_2)^{-\frac{1}{2}}$, T could be E_1 or E_3. In this case $T = E_1$ for $\rho_2 > \rho_1$, $T = E_3$ for $\rho_1 > \rho_2$. The various equality cases missed out of this enumeration are those where equilibria coincide.

Also note that the relative position of T and the maximum of the parabola P_i is determined by the sign of $\theta q_i - r_i$ — if $\theta q_i \geq r_i$ then the segment of P_i from S to T is monotone; if $\theta q_i < r_i$ then this segment has a turning point before reaching T.

Regions of interest. Now let $\tilde{\Sigma}$ denote the rectangle in the w-plane with two sides on the axes intersecting at 0, a side through E_1 parallel to $\{w_1 = 0\}$ and one through E_3 parallel to $\{w_2 = 0\}$. Let Σ denote the open convex subset of $\tilde{\Sigma}$ whose boundary comprises four straight-line segments from $\partial\tilde{\Sigma}$, a parabolic segment from P_1 joining E_1 to E_2 and a parabolic segment from P_2 joining E_2 to E_3. Thus the boundary of Σ always has four straight-line segments: in addition it has two parabolic components when E_+, E_- and T are distinct, one parabolic component when two of E_+, E_- and T coincide and no parabolic component when $\theta > (4\rho_1\rho_2)^{-\frac{1}{2}}$. Also Σ consists of the union of three sets: $\omega_2 = \Sigma \cap \Omega_1 \cap \Omega_2$, a relatively closed component ω_3 whose boundary intersects $\{w_1 = 0\}$ away from the origin and a relatively closed component ω_1 whose boundary intersects $\{w_2 = 0\}$ away from the origin. (See Figure 5-1.) Note that

$$\begin{aligned} (r_1 + \theta q_1) w_1 - r_1 w_1^2 - \theta q_1 w_2 &> 0, \quad (w_1, w_2) \in \omega_2 \cup \omega_1, \\ (r_2 + \theta q_2) w_2 - r_2 w_2^2 - \theta q_2 w_1 &> 0, \quad (w_1, w_2) \in \omega_2 \cup \omega_3; \end{aligned}$$

and hence, if $w = (w_1, w_2)$ satisfies (2.2) then,

$$(b_1 + c) w_1' \text{ is strictly positive at } x \text{ if } w(x) \in \omega_2 \cup \omega_1,$$

$$
\begin{aligned}
&(b_2+c)w_2' \text{ is strictly positive at } x \text{ if } w(x) \in \omega_2 \cup \omega_3, \\
&(b_1+c)w_1' \text{ is strictly negative at } x \text{ if } w(x) \in \omega_3 \setminus \overline{\omega}_2, \\
&(b_2+c)w_2' \text{ is strictly negative at } x \text{ if } w(x) \in \omega_1 \setminus \overline{\omega}_2.
\end{aligned}
\tag{5.1}
$$

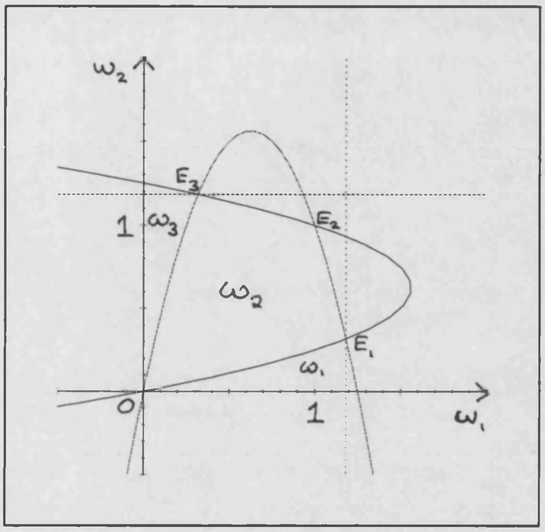


Figure 5-1: The regions of interest

A monotone curve connecting S to E_1 must approach E_1 from $\omega_1 \cup \omega_2$. Thus, by (5.1), if $(b_1+c) < 0$ there cannot be a monotone connection from S to E_1. Similarly, a monotone curve connecting S to E_3 must approach E_3 from $\omega_2 \cup \omega_3$. Thus, by (5.1), if $(b_2+c) < 0$ there cannot be a monotone connection from S to E_1.

A monotone connection to E_2 must eventually approach through ω_2, so a necessary condition for existence is $(b_1+c) > 0$ *and* $(b_2+c) > 0$.

When considering monotone connections from S to T it is sufficient to restrict attention to the intersections of each ω_i with the unit square — a monotone connection to T must lie entirely within the unit square and must eventually approach T through some ω_i. It is possible to work with just ω_2 and ω_3 — since ω_1 is mapped to ω_3 by swapping b_1 and b_2, r_1 and r_2 and q_1 and q_2. (It is not possible for a monotone connection from S to T to lie partly within ω_1 and partly within ω_3 since both w_1' and w_2' have signs opposite in ω_1 to their signs in ω_3.) The *outside* region is ω_3 intersected with the unit square; the *inside* region is a subset of ω_2 inside the unit square. Examination of these two regions allows us to cover all possible monotone connections to T — connections must eventually approach from one of the two.

Also note that the nullclines represent points at which solution curves are vertical or horizontal (where the nullclines cross, there are fixed points). Thus, if a solution curve hits a nullcline somewhere other than an equilibria it immediately crosses to the other side of the nullcline, tangency is not possible since the nullclines are nowhere vertical (for the one representing vertical solution curves) and nowhere horizontal (for the other).

5.3 The inside region

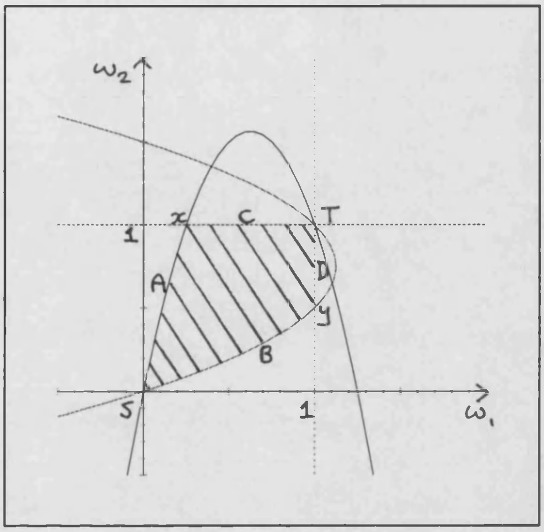


Figure 5-2: An example of an inside region with 4 distinct edges

Consider the region formed by the segment of the parabola P_1 from S to the first intersection with the line $w_2 = 1$ (this is at T if $\frac{\theta q_1}{r_1} \leq 1$); if the intersection is not at T then use the line $w_2 = 1$ to connect the endpoint of the segment to T; the segment of P_2 from S to the first intersection with the line $w_1 = 1$ (this is at T if $\frac{\theta q_2}{r_2} \leq 1$) and if the intersection is not at T then use the line $w_1 = 1$ to connect the endpoint to T. Thus, the *inside region* is defined to be the region *inside* both parabolae and the unit square, in the notation of the previous section it is a subset of ω_2. See Figures 5-2 and 5-3.

Denote the edges of the region as follows:

- The open segment of P_1 (not including S or the other endpoint) by A.
- The open segment of P_2 (not including S or the other endpoint) by B.
- If A does not connect S to T then denote the open segment of $w_2 = 1$ (not including endpoints) by C.
- If B does not connect S to T then denote the open segment of $w_1 = 1$ (not including endpoints) by D.

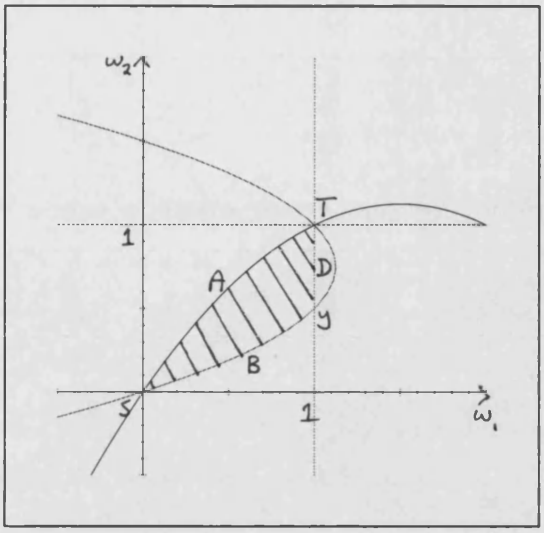


Figure 5-3: An example of an inside region with only 3 distinct edges

Notice that a monotone connection from S to T must approach T from inside this region if both C and D exist. If either C or D does not exist then it is possible to reach T monotonically from outside the *inside region* — then we must use the *outside* region as discussed below.

If C exists, then denote its left endpoint by x and if D exists then denote its lower endpoint by y.

By (5.1) and the following discussion, a necessary condition for a monotone connection that eventually approaches T from the inside region is that $c > \max(-b_1, -b_2)$, i.e.

$$b_1 + c > 0 \qquad \textit{and} \qquad b_2 + c > 0, \qquad \text{(C1)}$$

since $w_1' > 0$ and $w_2' > 0$ (throughout the interior of the region) if and only if (C1) holds.

Lemma 5.1 *Assume (C1) holds. Then a solution curve that hits the boundary of the inside region at any point other than S or T immediately crosses the boundary.*

Proof. Consider the various segments of the boundary. On A and at the point x, if it exists, $w_1' = 0$ and $w_2' > 0$. Thus (using continuity and the fact that the slope of this boundary is bounded and positive, so the flow is not tangent to the boundary) if $w(t) \in A$ or $w(t) = x$ then there exists an $\epsilon > 0$ such that for $s \in (t-\epsilon, t)$, $w(s)$ is inside the region and for $s \in (t, t+\epsilon)$, $w(s)$ is outside the region.

Similarly for B and y — $w_2' = 0$ and $w_1' > 0$ so the same argument works.

For C, if it exists, note that $w_1' > 0$ and $w_2' > 0$ on C and again the solution curve crosses from inside to outside automatically. Similarly for D, and the lemma is proved. □

5.4 The outside region

We have observed that there are monotone curves from S to T that do not intersect the inside region, except at T itself, when the inside region does not have four edges. Without loss of generality assume that the edge C of the inside region does not exist, i.e. $\frac{\theta q_1}{r_1} > 1$, so that there is a possibility of a monotone connection through ω_3. When neither C nor D exists there are two outside regions, by equation (5.1) the flow is in opposite directions in the two regions. Hence there can be a monotone connection through at most one of them.

Consider the region formed by the segment of the parabola P_1 from S to T; the w_2 axis and the line $w_2 = m(w_1 - 1) + 1$ for some $m \in (0, 1 - \frac{r_1}{\theta q_1})$. This restriction on m ensures that the line has positive slope and lies above the parabola P_1 (the upper bound just given is the slope of the parabola at T and the assumption that $\frac{\theta q_1}{r_1} > 1$ ensures that the slope is strictly positive). We will choose a particular value of m later. In the notation of section 5.2 this region is therefore a subset of ω_3. See Figure 5-4 for a diagram of a typical example of this region.

Denote the edges of the region as follows:

- The open segment of P_1 (not including S or T) by A.
- The half-open segment of the w_2 axis (not including S but including the other endpoint) by B.
- The open segment of $w_2 = m(w_1 - 1) + 1$ strictly between the w_2 axis and T by C.

For a monotone connection from S to T that does not go through the inside region it is necessary that $w_1' > 0$ and $w_2' > 0$ throughout the interior of the outside region (note that though there are curves from S to T that exit the outside region through C and still converge monotonically to T this condition will still be necessary).

Using equation (5.1) to consider the direction of the flow within the region notice that is is therefore necessary that $b_2 > b_1$ and that $-b_2 < c < -b_1$, i.e. $(b_2 + c) > 0 > (b_1 + c)$. So, for there to be a possibility of a monotone connection that is not through the inside region, the following conditions are necessary:

$$(b_2 + c) > 0 > (b_1 + c) \qquad \textit{and} \qquad \frac{\theta q_1}{r_1} > 1. \tag{C2}$$

For a monotone connection through the other outside region it is necessary that $(b_1 + c) > 0 > (b_2 + c)$ *and* $\frac{\theta q_2}{r_2} > 1$ — the subsequent analysis is entirely equivalent since we can simply interchange the subscripts 1 and 2. The two cases are clearly mutually exclusive.

The condition (C2) ensures that, for each $k > 0$, a curve that satisfies $w_2'/w_1' = k$ is an ellipse passing through S and T. This observation is used below to rule out internal tangencies to the region.

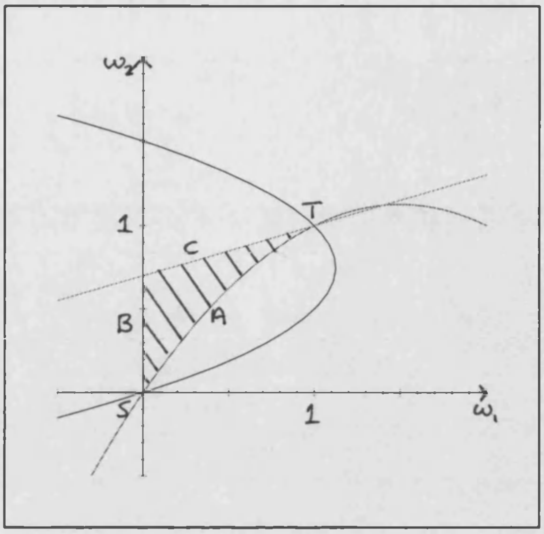


Figure 5-4: An example of an outside region

A solution curve cannot exit from the outside region through A or B in forwards time. On A observe that $w_1' = 0$ and $w_2' > 0$ and on B that $w_1' > 0$ and $w_2' > 0$.

We need to check that the flow does not have an internal tangency to C in order to construct a shooting argument for the outside region. The shooting argument used runs backwards in time, but note that a tangency backwards in time is also a tangency forwards in time, and vice versa.

Lemma 5.2 *Assume condition (C2) holds. Then a solution curve that hits the boundary of the outside region (from the inside of the region — an external tangency is immaterial) at any point other than S or T immediately crosses the boundary.*

Proof. If a solution curve hits A or B the observations above show that the curve (in backwards time) will cross from the inside to the outside of the region. So assume, for a contradiction, that at a point x_1 on C the solution curve is internally tangent to C. The solution curve in forwards time will re-enter the interior of the region — where $w_1' > 0$ and $w_2' > 0$ — and must eventually hit the line C again or else approach T. This is since it cannot leave the region through A because $w_1' = 0, w_2' > 0$ there. Call the first point at which this occurs x_2.

Thus, the solution curve is a smooth curve from x_1 to x_2, both of which are on the line $w_2 = m(w_1 - 1) + 1$, therefore there is a point x_3 on this curve at which the curve has slope m. However, the curve on which the slope of the flow is m is an ellipse; an ellipse that passes through S, T and x_1 — so x_3 cannot be on this ellipse. This contradiction proves the result. □

5.5 The shooting argument

To establish the main result (Theorem 2.1) we shall use the preceding observations in a shooting argument through the outside region (backwards in time), as well as a more standard forward time argument from S to T for the inside region. Note that a monotone travelling wave from S to T is possible only if T has stable monotone eigenvalues. A corresponding unstable direction at S is also necessary, but this always exists when a stable monotone eigenvalue at T exists. (See the discussion before Lemma 4.2 in Chapter 4.) It is known so far that if $c > \max(-b_1, -b_2)$ then a monotone connection can only exist through the inside region — we show below that in this case it does indeed exist and is unique. For $c < \min(-b_1, -b_2)$ there cannot be a monotone connection from S to T and for c in between then there may or may not be a monotone connection (which necessarily lies entirely in an outside region) — the determining factor is whether c is greater than $c(\theta)$. When such a connection exists it is unique.

5.5.1 Through the inside region when (C1) holds

Consider the intersection of the inside region with the circle of radius ϵ, centred at the origin, S. This is a segment of a circle with 2 endpoints on the boundary of the inside region. $w_1' > 0$ *and* $w_2' > 0$ are necessary for a monotone connection through the region (i.e. $(b_1 + c) > 0$ *and* $(b_2 + c) > 0$ are necessary in this region) — when this is true a connection looks plausible. Considering the family of solution curves which pass through points of this segment at time zero completes our shooting argument, since, running backwards in time all these curves must go to S (since they cannot exit the region and cannot tend to any point other than an equilibrium), and running forwards in time, the curves from the 2 endpoints leave the region immediately and curves from points in between will exit transversally from the region (by Lemma 5.1), thus exiting in between the exit/end-points. Hence there is a monotone connection between S and T.

5.5.2 Through the outside region when (C2) holds

Assume condition (C2), otherwise there is no monotone connection through the region. This time we shoot backwards in time.

Firstly we deal with $c > c(\theta)$. Choose m so that the edge C has slope equal to half that of the dominant eigenvector of $K_{c,\theta}(T)$ (the dominant eigenvector is that corresponding to the negative real eigenvalue of smallest modulus — when (C2) and $c > c(\theta)$ there are two simple stable monotone eigenvalues so m is well-defined). Thus the upper edge of the region bisects the angle between the line $w_2 = 1$ and the dominant eigenvector at T. This enables us, backwards in time, to obtain paths leaving the outside region on both sides of T, and hence shoot to S. More precisely, for $c > c(\theta)$ we already know that $K_{c,\theta}(T)$ has two stable monotone eigenvalues. Consideration of equation (4.2) shows that both have slope less than that of the P_1 at T. The dominant eigenvector is $\begin{pmatrix} 1 \\ v^+ \end{pmatrix}$. Thus we choose m to correspond to a direction

$\begin{pmatrix} 1 \\ v^+/2 \end{pmatrix}$. Hence, by shooting backwards from points suitably close to T and using the fact that the flow is determined by the dominant eigenvector we observe the flow sweeps through the boundaries of the region (by Lemma 5.2) as we follow a segment of a circle around T that intersects the region. Classical continuous dependence theory for initial value problems tells us that a connection from S to T exists.

For $c < c(\theta)$ there is no stable monotone eigenvalue at T and there cannot be a monotone connection. Thus we have completed showing that a monotone eigenvalue at T is sufficient (except for the cases $c = c(\theta)$ and $c = \max(-b_1, -b_2) > c(\theta)$ which are discussed below in section 5.6) for a monotone connection — we already knew it was necessary.

5.6 Special cases

We have been wary of the cases where $c = c(\theta)$ and $c = -b_i$. Assume, without loss of generality, that $b_1 > b_2$.

If $\theta q_2 \leq r_2$, then $c(\theta) = -b_2$. For $c = c(\theta)$ the only candidate for a connection from S to T is the segment of the w_2-nullcline, P_2, connecting S to T. If $\theta q_2 < r_2$ then this is not monotone from S to T and so there is no monotone travelling wave at this speed. When $\theta q_2 = r_2$ this connection is monotone. The travelling wave equations consists of one algebraic — defining the nullcline — and one differential equation. We can substitute the algebraic into the differential equation to obtain a one-dimensional problem where we are looking for a connection from 0 to 1. Since the derivative is positive between these points this indeed exists and the segment of nullcline is a monotone travelling wave.

When $\theta q_2 > r_2$, $-b_1 < c(\theta) < -b_2$. For the case $c = c(\theta)$ we can use the argument we used in the outside region — by part (iii) of Lemma 4.2 there is a double stable monotone eigenvalue so the corresponding eigenvector determines the nature of the flow near T. We simply repeat the bisection procedure and show that the flow leaves either side of the region — so there is a monotone connection for $c = c(\theta)$ in this case. We also must check the case $c = -b_2$, but here the solution is simply the segment of nullcline exactly as for $\theta q_2 = r_2$.

Thus we can summarise as follows:

- For $\theta q_2 < r_2$, there is a monotone travelling wave if and only if $c > c(\theta)$.
- For $\theta q_2 \geq r_2$. There is a monotone travelling wave if and only if $c \geq c(\theta)$.

5.7 Uniqueness modulo translation

To establish uniqueness of monotone travelling waves from S to E_i, $i = 1, 2, 3$, it suffices, because of the change of variables at (1.26), to prove that the wave from S to T is unique. However, due to the dimensionality of the unstable and stable manifolds at S and T respectively, we have very little left to show. As we summarised in Chapter 4; just above Lemma 4.2; we have that:

- For $c > \max(-b_1, -b_2)$; $K_{c,\theta}(T)$ has a one-dimensional stable manifold for $\theta \geq \frac{1}{\rho_1+\rho_2}$.
- For $c < \min(-b_1, -b_2)$; $K_{c,\theta}(S)$ has a two-dimensional *stable* manifold. (Of course, here $c < c(\theta)$ anyway.)
- For $\min(-b_1, -b_2) < c < \max(-b_1, -b_2)$; $K_{c,\theta}(S)$ has a one-dimensional unstable manifold.

Thus, since we can only possibly have non-uniqueness when:

- $K_{c,\theta}(S)$ has a two-dimensional unstable manifold *and* $K_{c,\theta}(T)$ has a two-dimensional stable manifold,

we have only one case left to check, i.e. $c > \max(-b_1, -b_2)$ and $\theta < \frac{1}{\rho_1+\rho_2}$.

In this case the two eigenvalues of T are negative and distinct, denote them by $\alpha < \beta < 0$, and the corresponding eigenvectors by v_α and v_β, respectively. α is a stable monotone eigenvalue (i.e. v_α has both components of the same sign) and β is not. Note that the dominant eigenvalue β is the non-monotone eigenvalue. This suggests that most of the flow entering T will do so non-monotonically and so there may indeed be a unique monotone connection. We now use this idea to complete the proof of uniqueness.

Firstly, by standard theory (Coddington and Levinson [18, Chapter 13, Theorem 4.4]), all solutions that converge to T do so exponentially. For notational convenience define u by $w_i(x) = u_i(x) + 1$ for $i = \{1, 2\}$, so that u converges to the origin when w converges to T. Then there exists a one-dimensional manifold of solutions u such that

$$\frac{\log \| u(x) \|}{x} \longrightarrow \alpha \text{ as } x \longrightarrow \infty.$$

and all other solutions that converge to the origin do so with rate β.

By Coddington and Levinson [18, Chapter 13, Theorem 4.5] it follows that these other solutions are of the form $u(x) = k_\beta e^{\beta x} v_\beta + o(e^{\beta x})$ as $x \to \infty$ for some $k_\beta \neq 0$. However, since v_β has components of opposite sign these are not monotone solutions, leaving us with only the one-dimensional manifold of solutions as candidates for monotone connections.

Hence uniqueness is proven and we have completed the proof of Theorem 2.1.

Chapter 6

Probabilistic results

6.1 A one-particle martingale

We start by considering a one-particle martingale for motivation, though it is not needed for the main results of this chapter. It indicates how the various terms in the partial differential equation correspond to terms in the probabilistic model, as well as demonstrating where the eigenproblems fit into the probability. In Champneys et al. [13] a one-particle martingale was needed to show that their Z_λ martingale was a *true* martingale, not just a *local* martingale. For this problem that can be shown directly — this is done at the end of section 6.2. Good references for this chapter are the two volumes of Rogers and Williams [61] and [62], and Ethier and Kurtz [28]. The large deviations material is covered by Varadhan [71] and the references therein.

Recall our notation that $I = \{1, 2\}$ and that we often switch between two equivalent notations in this chapter to reduce the use of subscripts. Thus we will sometimes write, for example, $b(y)$ for b_y, $w(x, y)$ for $w_y(x)$ and $u(t, x, y)$ for $u_y(t, x)$ (for $y = 1, 2$). We consider a process (ξ, η) on $\mathbb{R} \times I$, where η is an autonomous Markov chain with Q-matrix θQ, and where, while $\eta = y \in I$, ξ moves at constant velocity $b(y)$. Thus, (ξ, η) has formal generator $\mathcal{H}$, where

$$(\mathcal{H}F)(x, y) = \theta \sum_{j \in I} Q(y, j) F(x, j) + b(y) \frac{\partial F}{\partial x}.$$

Recall the definitions of B and R in equation (2.1), let $\lambda < 0$, let $\Lambda_{PF}(\lambda)$ be the Perron-Frobenius eigenvalue of $\lambda B + \theta Q + R$, and let v_λ be the associated eigenvector with $v_\lambda(1) = 1$. Suppose that the particle starts at position $x \in \mathbb{R}$ with type $y \in I$. Define

$$\zeta_\lambda(t) := \exp\left(\int_0^t r(\eta_s)\, ds \right) v_\lambda(\eta_t) \exp\bigl(\lambda \xi_t - \Lambda_{PF}(\lambda) t\bigr). \tag{6.1}$$

Now $\xi_t = x + \int_0^t b(\eta_s)\,ds$, so we can rewrite equation (6.1) as

$$\zeta_\lambda(t) = \exp\left(\int_0^t \big(r(\eta_s) + \lambda b(\eta_s)\big)\,ds\right) v_\lambda(\eta_t)\exp\big(\lambda x - \Lambda_{PF}(\lambda)t\big). \tag{6.2}$$

The study of martingales for a Markov chain with a finite state-space (just two states) is discussed in Rogers and Williams [62] section IV.20. That tells us that for every function f on I, where X_t is the Markov chain corresponding to a Q-matrix $\hat{Q}$,

$$f(X_t) - f(X_0) - \int_0^t (\hat{Q}f)(X_s)ds$$

is a local martingale. This shows us how to differentiate equation (6.2) using Itô's formula, and shows that ζ_λ is a local martingale, since we have arranged that

$$(R + \lambda B + \theta Q)v_\lambda - \Lambda_{PF}(\lambda)v_\lambda = 0.$$

Now ζ_λ is also non-negative, with $\zeta_\lambda(0) = v_\lambda(y)e^{\lambda x}$, so it is a supermartingale. Now to show that ζ_λ is a true martingale; write $b_0 = \min(0, b_1, b_2)$, $r_0 = \max(r_1, r_2)$, $v_\lambda(0) = \max\big(v_\lambda(1), v_\lambda(2)\big)$ to obtain

$$\sup_{s\le t}\zeta_\lambda(s) \le v_\lambda(0)\exp\Big(t\big(r_0 + \lambda b_0 - \Lambda_{PF}(\lambda)\big) + \lambda x\Big).$$

Hence $\sup_{s\le t}\zeta_\lambda(s)$ is in each $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}_{x,y})$, and hence ζ_λ is a true martingale.

6.2 Martingales for a branching two-type process

Consider the two-type branching system of particles, each of whose types and motions are as in section 6.1, which was defined in section 2.5.

The formal generator $\mathcal{G}$ of that process is

$$\mathcal{G} = \mathcal{G}_B + \mathcal{G}_Q + \mathcal{G}_R,$$

where, for $n \ge 1$, $x \in \mathbb{R}^n$, and $y \in I^n$, we have

$$\begin{aligned}
(\mathcal{G}_B F)(n;x;y) &= \sum_{k=1}^n b(y_k)\frac{\partial F}{\partial x_k},\\
(\mathcal{G}_Q F)(n;x;y) &= \theta\sum_{k=1}^n \sum_{j\ne y_k; j\in I} Q(y_k, j)\Big\{F\big(n;x;s_{k,j}(y)\big) - F\big(n;x;y\big)\Big\},\\
(\mathcal{G}_R F)(n;x;y) &= \sum_{k=1}^n r(y_k)\Big\{F\big(n+1;(x,x_k);(y,y_k)\big) - F\big(n;x;y\big)\Big\},
\end{aligned}$$

where $s_{k,j}(y) := (y_1, \ldots, y_{k-1}, j, y_{k+1}, \ldots, y_n)$ and $(x, x_k) := (x_1, \ldots, x_n, x_k)$, etc. If F :

$[0,\infty) \times \mathcal{S}$ (where $\mathcal{S}$ is the state-space of the branching process as defined by equation (2.7)) and

$$\left\{ \left(\frac{\partial}{\partial t} + \mathcal{G} \right) F \right\} (t; n; x; y) = 0 \qquad (n \geq 1,\ x \in \mathbb{R}^n,\ y \in I^n), \tag{6.3}$$

then $F\big(t; N(t); X(t); Y(t)\big)$ is a local martingale.

In particular, if u is a $C^{1,1}$ solution of the coupled system (2.1), define, for $t > 0$,

$$M(s) := \prod_{k=1}^{N(s)} u\big(t - s; X_k(s); Y_k(s)\big). \tag{6.4}$$

This satisfies the conditions of equation (6.3) (the partial differential equation enables us to do the time differentiation of M) and hence defines a 'multiplicative' local martingale M on the time-parameter set $[0,t]$. If in addition $0 \leq u(\cdot, x, y) \leq 1$, then $0 \leq M \leq 1$, so that M is a true martingale and

$$\mathbb{E}_{x,y} M(0) = \mathbb{E}_{x,y} M(t),$$

that is,

$$u(t,x,y) = \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} f\big(X_k(t), Y_k(t)\big) = \mathbb{E}_{0,y} \prod_{k=1}^{N(t)} f\big(x + X_k(t), Y_k(t)\big),$$

where $f(x,y) = u(0,x,y)$ is C^1.

Thus we have proved the following:

Theorem 6.1 *If $u \in C^{1,1}$ satisfies the system* (2.1) *with* $0 \leq u(t,x,y) \leq 1$ *on* $[0,\infty) \times \mathbb{R} \times I$ *and with initial condition*

$$u(0,x,y) = f(x,y) \in C^1,$$

then u has a McKean representation

$$u(t,x,y) = \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} f\big(X_k(t), Y_k(t)\big).$$

The above theorem clearly tells us that there is at most 1 bounded (between 0 and 1) solution to the system (2.1) for smooth, bounded (between 0 and 1) initial data.

Now we look at 'additive' martingales. If $h : [0,\infty) \times \mathbb{R} \times I \to \mathbb{R}$ satisfies the *linear* equation

$$\frac{\partial h}{\partial t} + B\frac{\partial h}{\partial x} + \theta Q h + R h = 0,$$

then, again by using (6.3),

$$\sum_{k=1}^{N(t)} h\big(t, X_k(t), Y_k(t)\big) \text{is a local martingale.}$$

Now we have:

Theorem 6.2 *If w is a C^1 function on $\mathbb{R} \times I$, then*

$$\prod_{k=1}^{N(s)} w(X_k(s) + cs, Y_k(s))$$

is a local martingale if, and only if, w solves the travelling-wave system (2.2).

If g is a C^1 function on $\mathbb{R} \times I$, then

$$\sum_{k=1}^{N(t)} g(X_k(t) + ct, Y_k(t))$$

is a local martingale if, and only if, g solves the linearization of (2.2) *at the point* (1, 1):

$$(B + c)g' + \theta Qg + Rg = 0.$$

This theorem follows from the fact that in (6.4) M is a local martingale and from the local-martingale property established immediately before Theorem 6.2. This can be seen by noting that we are using a travelling wave solution as follows, $u(t - s, x, y) = w(x - c(t - s), y) = w(x - ct + cs, y)$, and the ct term just shifts along the travelling wave as the variable being used here is s.

Let $c > c(\theta)$, where $c(\theta)$ is defined in Theorem 2.1. If $c < \max(-b_1, -b_2)$, take λ to be the stable monotone eigenvalue of $K_{c,\theta}(T)$ nearer to 0; if $c = \max(-b_1, -b_2)$, take λ to be the unique generalized stable monotone eigenvalue of $K_{c,\theta}(T)$ and if $c > \max(-b_1, -b_2)$, take λ to be the unique stable monotone eigenvalue of $K_{c,\theta}(T)$. This definition of λ is possible and well-defined by the the work at the start of Chapter 4. Call, for future reference, this definition of λ the *probabilistic* eigenvalue of $K_{c,\theta}(T)$. Hence, if λ is the *probabilistic* eigenvalue of $K_{c,\theta}(T)$ then $-\lambda c$ is the Perron-Frobenius eigenvalue of $\lambda B + \theta Q + R$. Recall the definition of Z_λ in Theorem 2.4,

$$Z_\lambda(t) = \sum_{k=1}^{N(t)} v_\lambda(Y_k(t)) \exp\left\{ \lambda[X_k(t) + ct] \right\}.$$

Then, by Theorem 6.2, Z_λ is an 'additive' local martingale; and since it is non-negative, it is also a supermartingale. We wish to show that Z_λ is a true martingale, this can be done just as for ζ_λ by noting that $\mathbb{E}N(t) \leq e^{r_0 t}$ and each individual term of the sum is again bounded up to any fixed t.

6.3 A large-deviations approach to $c(\theta)$

As we have already observed in Theorem 2.3, the asymptotic behaviour of the position of the left-most particle of the system should give us the wave speed $c(\theta)$. This is proved later in this chapter; for now we want to give probabilistic heuristics to obtain this value.

If we consider a particle's type intuitively we note that it flicks between the two types with an equilibrium distribution independent of θ. On average the particle will spend a proportion $\frac{q_2}{q_1+q_2}$ of its time with type 1 and a proportion $\frac{q_1}{q_1+q_2}$ of its time with type 2. A larger θ will tend to force each particle's type distribution closer to the equilibrium distribution.

So, we are interested in considering how far from the equilibrium distribution (for time spent in the two states) a particle can deviate. The largest deviations will give the positions of the most extreme particles, including the left-most particle. Clearly, the higher the breeding rate (the parameters r_1, r_2) for each state, the easier it will be to find a particle which has been in that state for an unusually high proportion of the time. As θ is increased the deviations from the equilibrium distribution will be smaller.

The theory of large deviations is the appropriate technique for tackling such a problem. We use the Dirichlet form $\epsilon(u, v) := -u^T \Pi Q v$, where Π is the diagonal matrix whose elements are the components of the equilibrium distribution for the Markov chain. This is a positive-definite, symmetric bilinear form. Then the large-deviation rate functional $J_\epsilon(\mu)$ is $\epsilon(f^{1/2}, f^{1/2})$ where f is the ratio of μ to the invariant distribution and μ is any distribution for time spent in the two states (see Deuschel and Stroock [23, page 129]). So, if μ corresponds to spending a proportion of time m_1 in state 1, then $J_\epsilon(\mu) = (\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2$, and hence enables us to state that the probability that a particle, at time t, has spent a proportion m_1 of its time in state 1 and a proportion m_2 of its time in state 2 (for $m_1, m_2 \geq 0, m_1 + m_2 = 1$) is approximately $\exp\bigl(-t\theta(\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2\bigr)$. Thus, for all proportions other than the equilibrium proportions this decays exponentially and an increase in θ increases the rate of decay. Now, this decay is balanced by the breeding — the number of particles grows exponentially at a rate $m_1 r_1 + m_2 r_2$. So, when

$$m_1 r_1 + m_2 r_2 - \theta(\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2 > 0 \tag{6.5}$$

there is exponential growth in the number of particles; when this is strictly less than zero there is decay. This leads to a conjecture that $c(\theta)$ can be defined by the following formula, we denote the defined quantity by $\tilde{c}(\theta)$ and then show that $c(\theta) = \tilde{c}(\theta)$:

$$\begin{aligned}\tilde{c}(\theta) = \quad -\inf\Big\{ b_1 m_1 + b_2 m_2 : \quad & m_1 r_1 + m_2 r_2 - \theta(\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2 > 0; \\ & m_1, m_2 \geq 0, m_1 + m_2 = 1 \Big\},\end{aligned}$$

where m_1 and m_2 represent the proportions of times in the two states. This is well-defined since for $m_1 = \frac{q_2}{q_1+q_2}$ (the equilibrium distribution) the inequality is certainly satisfied.

To demonstrate the correspondence of $c(\theta)$ and $\tilde{c}(\theta)$ we examine some specific cases.

Firstly if $b_1 = b_2 = b$ then the above formula gives $\tilde{c}(\theta) = -b$, which tallies with the result for $c(\theta)$ we obtained in Chapter 4. As before we can now assume, without loss of generality, that $b_1 > b_2$. For the equilibrium proportions the inequality (6.5) is satisfied, so we immediately have that $c^* = -(\frac{b_1 q_2 + b_2 q_1}{q_1 + q_2}) \leq \tilde{c}(\theta) \leq -b_2$.

So we ask, when is $\tilde{c}(\theta) = -b_2$? Putting $m_2 = 1$ the inequality (6.5) becomes $r_2 - \theta q_2 > 0$, so $r_2 > \theta q_2$ implies that $\tilde{c}(\theta) = -b_2$. For $r_2 = \theta q_2$ we use the concavity of $m_1 r_1 + m_2 r_2 -$

$\theta(\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2$ to observe that the inequality (6.5) is satisfied for $m_2 = 1 - \epsilon$ for all ϵ in the interval $0 < \epsilon < \frac{q_2}{q_1+q_2}$, thus the infimum is still $m_2 = 1$ — just it is not attained (this corresponds to the fact that in this case there is a travelling wave corresponding to $\tilde{c}(\theta)\big(= c(\theta)\big)$, for $r_2 > \theta q_2$ there is not).

For $r_2 < \theta q_2$, $\tilde{c}(\theta) < -b_2$ and so all that remains to prove is that $\tilde{c}(\theta) = c(\theta)$ in this case too. We can rearrange our formula for $\tilde{c}(\theta)$ by writing $c = -b_1 m_1 + b_2(m_1 - 1)$ and hence $m_1 = \frac{b_2+c}{b_2-b_1}$. Substituting this into the defining inequality we note that

$$m_1 r_1 + m_2 r_2 - \theta(\sqrt{q_1 m_1} - \sqrt{q_2 m_2})^2 = 0$$

implies that the term under the square root in equation (4.1) is zero. Now we note that this corresponds to the zero with the larger value of c since $\tilde{c}(\theta)$ is at least c^*.

Thus we have confirmed that large-deviation theory has motivated an equivalent expression for $c(\theta)$ to that which we obtained in Chapter 4.

We notice that this formula makes it easier to observe that we can have $m_i = 1$ only when $r_i > \theta q_i$, which makes intuitive sense — the breeding in state i needs to be larger than the mutation out of the state for there to be a persistent family of that type of particle over time. It also makes it clear that $c(\theta)$ decreases from $\max(-b_1, -b_2)$ to c^* as $\theta \to \infty$.

6.4 Convergence properties of Z_λ martingales

The full assumptions which we have made about Z_λ (defined in Theorem 2.4) — that $c > c(\theta)$, that λ is the *probabilistic* eigenvalue of $K_{c,\theta}$, etc. — will now be needed in proving that Z_λ converges in $\mathcal{L}^p$ for some $p > 1$ (and hence in $\mathcal{L}^1$). According to Doob's $\mathcal{L}^p$ inequality (see Rogers and Williams [61], Theorem II.70.2), we need only show that Z_λ is bounded in $\mathcal{L}^p$ for some $p > 1$.

For investigating convergence in $\mathcal{L}^p$, the following result is indispensable. The result is taken from Neveu [53], and the method of using it is taken from Champneys et al. [13].

Lemma 6.3 (Neveu) *Let $p \in (1, 2]$. For any finite sequence $W_1, \dots, W_n$ of non-negative independent variables in $\mathcal{L}^p$ and any sequence $c_1, \dots, c_n$ of non-negative real numbers,*

$$\psi\left(\sum_{k=1}^{n} c_k W_k\right) \le \sum_{k=1}^{n} c_k^p \psi(W_k),$$

where $\psi(W) := \mathbb{E}(W^p) - \big(\mathbb{E}(W)\big)^p$ for $W \in \mathcal{L}^p$.

Proof of $\mathcal{L}^1$ convergence of Z_λ. Fix $t > 0$. Because of the branching character of the (N, X, Y) process, we have for each $s > 0$,

$$Z_\lambda(s+t) = \sum_{k=1}^{N(s)} \exp\{\lambda\,[X_k(s) + cs]\}\, W_k(t, s)$$

where, conditionally on $\mathcal{F}_s$, the $W_k(t,s)$ are independent, each with the $\mathbb{P}_{0,y(k)}$ law of $Z_\lambda(t)$ where $y(k) = Y_k(s) \in I$. Since t is fixed, and I is finite, Neveu's lemma, applied conditionally on $\mathcal{F}_s$, gives

$$\mathbb{E}_{x,y}\left\{Z_\lambda(s+t)^p \mid \mathcal{F}_s\right\} - Z_\lambda(s)^p \leq K_1(t,x,\lambda) \sum_{k=1}^{N(s)} \exp\{\lambda p\,[X_k(s)+cs]\},$$

so that, on taking expectations,

$$\mathbb{E}_{x,y}\left\{Z_\lambda(s+t)^p\right\} - \mathbb{E}_{x,y}\left\{Z_\lambda(s)^p\right\} \leq K_1(t,x,\lambda)\mathbb{E}_{x,y}\left\{\sum_{k=1}^{N(s)} \exp\{\mu\,[X_k(s)+cs]\}\right\},$$

where $\mu := \lambda p < \lambda < 0$. We choose p in $(1,2]$ sufficiently close to 1 that (see Lemma 4.2) $c > c_1$, where $-\mu c_1 = \Lambda_{PF}(\mu B + \theta Q + R)$ and v_μ is the corresponding eigenvector with $v_\mu(1) = 1$. But then

$$\begin{aligned} & \mathbb{E}_{x,y} \sum_{k=1}^{N(s)} \exp\{\mu\,[X_k(s)+cs]\} \\ \leq\; & K_2(x,\mu,y)\exp\{\mu(c-c_1)s\}\mathbb{E}_{x,y}\sum_{k=1}^{N(s)} v_\mu(Y_k(s))\exp\{\mu\,[X_k(s)+c_1 s]\} \\ \leq\; & K_3(x,\mu,y)\exp\{\mu(c-c_1)s\}, \end{aligned}$$

since Z_μ is a martingale. Hence,

$$\begin{aligned} & \sum_m \mathbb{E}_{x,y}\left\{Z_\lambda(ms+s)^p - Z_\lambda(ms)^p\right\} \\ \leq\; & K_1(s,x,\lambda)K_3(x,\mu,y)\sum_m \exp\{\mu m(c-c_1)s\} < \infty, \end{aligned}$$

and Z_λ is bounded in $\mathcal{L}^p$. □

Now we prove, with the same notation and assumptions, that

$$w(y) := \mathbb{P}_{x,y}\big(Z_\lambda(\infty) = 0\big) = 0 \text{ for all } (x,y). \tag{6.6}$$

(The fact that $w(y)$ does not depend on x is obvious.)

Proof of (6.6). Let J be the first jump time of Y_1 and let T be the first branch time of (N,X,Y). On decomposing $w(y)$ according as $T < J$ or $T > J$, we obtain

$$w(y) = \frac{r(y)w(y)^2 + \theta\sum_{z\neq y} Q(y,z)w(z)}{r(y)+\theta q(y)},$$

so that $Rw = R(w^2) + \theta Q w$. By Lemma 4.4, $w \equiv 0$ on I or $w \equiv 1$ on I. When Z_λ converges in $\mathcal{L}^1$, then, obviously, $w \equiv 0$ on I. $\square$

The following theorem is now proven.

Theorem 6.4 *Let $c > c(\theta)$. Let λ be the* probabilistic *eigenvalue of $K_{c,\theta}(T)$. Thus $-\lambda c$ is the Perron-Frobenius eigenvalue of $\lambda B + \theta Q + R$ and, as usual, we denote by v_λ the corresponding eigenvector with $v_\lambda(1) = 1$. Then*

$$Z_\lambda(t) = \sum_{k=1}^{N(t)} v_\lambda\big(Y_k(t)\big) \exp\big\{\lambda\,[X_k(t) + ct]\big\}.$$

is a true (not just a local) martingale, and $Z_\lambda(t)$ converges to a limit $Z_\lambda(\infty)$ almost surely and in $\mathcal{L}^1$. Moreover, $\mathbb{P}_{x,y}(Z_\lambda(\infty) > 0) = 1$ for all x and y.

For the proof of the uniqueness modulo translation of the monotone travelling wave from S to T, we need the following result. When there is only one stable monotone eigenvalue of $K_{c,\theta}(T)$ we have no other candidate for a wave (each stable monotone eigenvalue gives us a martingale which could correspond to a suitable wave), when there are two we use this result to rule out the *other* possibility.

Lemma 6.5 *Suppose that $c > c(\theta)$ and that there are two stable monotone eigenvalues of $K_{c,\theta}(T)$. Denote by β the eigenvalue further from 0, and by v_β the associated Perron-Frobenius eigenvector of $\beta B + \theta Q + R$ with $v_\beta(1) = 1$. Then, almost surely,*

$$Z_\beta(t) = \sum_{k=1}^{N(t)} v_\beta\big(Y_k(t)\big) \exp\big\{\beta\,[X_k(t) + ct]\big\} \to 0$$

as $t \to \infty$.

Proof. We prove this result by modifying an argument in Neveu [53]. Let $0 < p < 1$. Then for $u, v > 0$,

$$(u+v)^p \le u^p + v^p.$$

Again, let J be the first jump time of Y_1 and let T be the first branch time of (N, X, Y). The decomposition

$$Z_\beta(\infty) = \begin{cases} \exp\{\beta[X_1(J) + cJ]\}\, Z_\beta^{(1)}(\infty) & \text{if } J < T, \\ \exp\{\beta[X_1(T) + cT]\}\,[Z_\beta^{(2)}(\infty) + Z_\beta^{(3)}(\infty)] & \text{if } T < J, \end{cases}$$

leads to the formula

$$\begin{aligned} g(y) := \mathbb{E}_{0,y}\,[Z_\beta(\infty)^p] &\le \mathbb{E}_{0,y} \exp\{\alpha[X_1(J) + cJ]\} I_{J<T}\, g\big(Y_1(J)\big) \\ &+ 2\mathbb{E}_{0,y} \exp\{\alpha[X_1(T) + cT]\} I_{T<J}\, g\big(Y_1(T)\big), \end{aligned}$$

where $\alpha := p\beta$. On evaluating these expectations, we obtain

$$g(y) \leq \frac{\left\{\theta \sum_{z \neq y} Q(y,z)g(z)\right\} + 2r(y)g(y)}{-\left(b(y)+c\right)\alpha + r(y) + \theta q(y)},$$

which rearranges to give

$$0 \leq \left((B + cI)\alpha + \theta Q + R\right)g.$$

We know that $g \geq 0$ on I. Lemma 4.3 shows that if we choose p sufficiently close to 1, then $g = 0$. The lemma is proved. □

6.5 Proof of Theorem 2.4

We now verify steps in the proof of Theorem 2.4, which lead to the important equality (2.8) of Theorem 2.3:

$$\lim t^{-1}L(t) = -c(\theta) \quad \text{(a.s.)} \quad \text{where } L(t) = \inf_{k \leq N(t)} X_k(t).$$

Part (i). Let $c > c(\theta)$, and let λ be the *probabilistic* eigenvalue of $K_{c,\theta}(T)$. Let Z_λ be the associated martingale. We see by considering the position of the left-most particle that

$$Z_\lambda(t) = \sum_{k=1}^{N(t)} v_\lambda\left(Y_k(t)\right)\exp\left\{\lambda\left[X_k(t)+ct\right]\right\} \geq \min\left(v_\lambda(1), v_\lambda(2)\right)\exp\left\{\lambda\left[L(t)+ct\right]\right\}.$$

Since $Z_\lambda(\infty)$ exists a.s., $\liminf[L(t)+ct] > -\infty$, a.s., so that

$$\liminf t^{-1}L(t) \geq -c = \lambda^{-1}\Lambda_{PF}(\lambda).$$

Part (ii). Before Theorem 2.4 was stated we noted that we would consider working with the $\mathbb{P} = \mathbb{P}_{0,1}$ law, so that $Z_\lambda(0) = 1$. Since Z_λ converges in $\mathcal{L}^1$, we can define a probability measure Q_λ on $\mathcal{F}_\infty$ via

$$dQ_\lambda/d\mathbb{P} = Z_\lambda(\infty) \text{ on } \mathcal{F}_\infty, \qquad \text{whence } dQ_\lambda/d\mathbb{P} = Z_\lambda(t) \text{ on } \mathcal{F}_t.$$

Define

$$M_\lambda(t) := Z_\lambda(t)^{-1}\frac{\partial}{\partial\lambda}Z_\lambda(t).$$

Because $(\partial/\partial\lambda)Z_\lambda(t)$ is a $\mathbb{P}$-martingale, M_λ is a Q_λ-martingale.

For $t \geq 0$ and $1 \leq k \leq N(t)$, define

$$H(t,k) := \frac{v_\lambda\left(Y_k(t)\right)\exp\left[\lambda X_k(t) - \Lambda_{PF}(\lambda)t\right]}{\sum_{j=1}^{N(t)} v_\lambda\left(Y_j(t)\right)\exp\left[\lambda X_j(t) - \Lambda_{PF}(\lambda)t\right]}.$$

Note that $H(t,k) \geq 0$ and $\sum_j H(t,j) = 1$. Now (denoting by prime differentiation with respect to λ),

$$M_\lambda(t) = \sum_{k=1}^{N(t)} H(t,k)\Big\{u_\lambda(Y_k(t)) + X_k(t) - \Lambda'_{PF}(\lambda)t\Big\},$$

where $u_\lambda(j) := v'_\lambda(j)/v_\lambda(j)$, so that

$$t^{-1}M_\lambda(t) \geq t^{-1}\{\min_{i\in I} u_\lambda(i)\} + t^{-1}L(t) - \Lambda'_{PF}(\lambda). \tag{6.7}$$

By Jensen's inequality,

$$M_\lambda(t)^2 \leq \sum H(t,k)\Big\{u_\lambda(Y_k(t)) + X_k(t) - \Lambda'_{PF}(\lambda)t\Big\}^2. \tag{6.8}$$

However,

$$Z_\lambda(t)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_\lambda(t)$$

is a Q_λ-martingale, and clearly

$$\begin{aligned} Z_\lambda(t)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_\lambda(t) &= \sum H(t,k)\Big\{u_\lambda(Y_k(t)) + X_k(t) - \Lambda'_{PF}(\lambda)t\Big\}^2 \\ &+ \sum H(t,k)\Big\{u'_\lambda(Y_k(t)) - \Lambda''_{PF}(\lambda)t\Big\}. \end{aligned}$$

Thus, (6.8) shows that the Q_λ-expectation of $M_\lambda(t)^2$ satisfies the inequality

$$Q_\lambda\left[M_\lambda(t)^2\right] \leq Q_\lambda\left[Z_\lambda(t)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_\lambda(t)\right] - Q_\lambda\left[\sum H(t,k)\Big\{u'_\lambda(Y_k(t)) - \Lambda''_{PF}(\lambda)t\Big\}\right].$$

The first term on the right-hand side is constant by the martingale property so that

$$Q_\lambda\left[M_\lambda(t)^2\right] \leq K_1(\lambda) - Q_\lambda\left[\sum H(t,k)\Big\{\max(u'_\lambda(1), u'_\lambda(2)) - \Lambda''_{PF}(\lambda)t\Big\}\right].$$

Since $\sum_j H(t,j) = 1$ further simplification gives

$$Q_\lambda[M_\lambda(t)^2] \leq K_2(\lambda) + K_3(\lambda)t$$

for finite constants $K_1(\lambda)$, $K_2(\lambda)$ and $K_3(\lambda)$, independent of t. Hence, for $\epsilon > 0$,

$$Q_\lambda\Big(\sup\left\{s^{-1}|M_\lambda(s)| : 2^{n-1} \leq s \leq 2^n\right\} \geq \epsilon\Big)$$

$$\leq Q_\lambda\Big(\sup_{s\leq 2^n}|M_\lambda(s)| \geq \epsilon 2^{n-1}\Big) \leq \left(\epsilon 2^{n-1}\right)^{-2}\left[K_2(\lambda) + 2^n K_3(\lambda)\right],$$

by Doob's submartingale inequality. By the Borel-Cantelli lemma, we have $t^{-1}M_\lambda(t) \to 0$, a.s., whence, from (6.7),

$$\limsup_{t\to\infty} t^{-1}L(t) \le \Lambda'_{PF}(\lambda).$$

Part (ii) of Lemma 4.2 clinches Part (iii) of Theorem 2.4, and the proof of (2.8) is complete. □

6.6 Heaviside initial conditions

We have the McKean representation

$$u(t,x,y) = \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} u\big(0, X_k(t), Y_k(t)\big)$$

for the unique (by the work of Chapter 3) solution of our coupled equation (2.1) when $0 \le u \le 1$ and the initial data $u(0,\cdot,\cdot)$ are sufficiently smooth. We would like to obtain a McKean representation in the case of the Heaviside initial data. We verify this directly below then, because $t^{-1}L(t) \to -c(\theta)$ (a.s.), the rest of Theorem 2.3 is obvious. In Chapter 7 we

- give a new direct analytic method of proving existence and uniqueness of *weak* solutions with bounded, measurable initial data,
- show that these solutions lead to martingales and hence to a McKean representation, and
- study the long-term behaviour of these *Heaviside* solutions.

Chapter 8 includes numerical and simulation studies of the solutions with Heaviside initial data.

We now verify that, *defining* $u(t,x,y)$ to be $\mathbb{P}_{x,y}[L(t)>0]$, gives a *weak* solution of the coupled equation (2.1) with Heaviside initial data. Conditioning on the first jump

$$u(t,x,y) = \exp\Big\{-\big(r(y)+\theta q(y)\big)t\Big\}u\big(0, x+b(y)t, y\big)$$

$$+\int_{s=0}^{t}\Big\{\exp\Big(-\big(r(y)+\theta q(y)\big)s\Big)\Big\}\Big\{r(y)u^2\big(t-s, x+b(y)s, y\big) + \theta q(y)u\big(t-s, x+b(y)s, \hat{y}\big)\Big\}ds$$

where $\{y,\hat{y}\} = \{1,2\}$. Integrating by parts and comparing the terms with those obtained by differentiating this expression for $u(t,x,y)$ by each of t and x yields

$$u(t,x,y) = \frac{r(y)}{r(y)+\theta q(y)}u^2(t,x,y) + \frac{\theta q(y)}{r(y)+\theta q(y)}u(t,x,\hat{y}) + \frac{b(y)\partial_x - \partial_t}{r(y)+\theta q(y)}u(t,x,y)$$

hence $u(t,x,y)$ satisfies equation (2.1). Clearly this expression for u corresponds to the Heaviside initial conditions.

We can observe several features of the solution from this representation. Since the left-most particle must lie in the interval $[x+t\min(b_1,b_2), x+t\max(b_1,b_2)]$ at time t then the solution is identically zero for $x \le -t\max(b_1,b_2)$ and is identically one for $x > -t\min(b_1,b_2)$. It is clear that the solution retains the left-continuity of the initial data. Also, since the left-most particle has a chance of being in any interval in this region (the left-most particle's position has a non-zero probability density function throughout this interval), $u(t, x, y)$ must be strictly increasing (in x) inside this region.

The case $b_1 = b_2 = b$, say, is very simple. All particles will travel at speed b, so that, at time t, all particles will be at $x + bt$. Hence, $L(t) = x + bt$, so $\mathbb{P}_{x,y}[L(t) > 0] = I_{\{x>-bt\}}$. This is simply the Heaviside function travelling at speed $-b$. This corresponds to the analysis in section 3.8.

To deal with $b_1 \neq b_2$ assume from now on, without loss of generality, that $b_1 > b_2$.

Note that the only way that the left-most particle can be at position $x + b_1 t$ at time $t > 0$ is if the first particle was of type 1 (otherwise there will have been a finite length of time for which the first particle travelled at b_2 and after that the subsequent family of particles can never catch up) and there have been no mutations to type 2 by the particle or any of its descendants so far. The probability of this event occurring when the initial particle is of type 1 is

$$\frac{r_1 + \theta q_1}{r_1 + \theta q_1 \exp\big((r_1 + \theta q_1)t\big)},$$

which explains why $u(t, -t+)$ equals this in the calculation in section 3.6.

The only way that the left-most particle can be at position $x + b_2 t$ at time $t > 0$ is if the first particle was of type 2 (otherwise there will have been a finite length of time for which the first particle travelled at b_1 and after that the subsequent family of particles will always be ahead of this position) and the embedded birth and death process on this line has not become extinct. This embedded process is that constructed by considering a particle to die when it mutates to type 2. Thus, on $x = -b_2 t$, for $t > 0$, u_1 is identically zero. The probability of this event occurring when the initial particle is of type 2 is exactly the expression calculated for $v(t, +t)$ in section 3.6, and the limit as $t \to \infty$ (which is $\min(\frac{\theta q_2}{r_2}, 1)$) is the probability of eventual extinction.

More generally it is natural to expect that the McKean representation will tie up with *weak* solutions of our coupled equation (2.1). This is because they are piecewise classical and any discontinuities (corresponding to atoms of probability in the distribution of the particles' positions) will naturally propagate only along characteristics (since the atoms of probability travel at b_1 and b_2). Thus a solution satisfying the McKean representation satisfies the Rankine-Hugoniot conditions, hence is a *weak* solution (see section 3.4).

6.7 Polishing off the probability

Theorem 6.6 *Assume $c > c(\theta)$, let λ be the* probabilistic *eigenvalue of $K_{c,\theta}(T)$, and as usual denote by v_λ be the corresponding eigenvector with $v_\lambda(1) = 1$. Then if u satisfies the coupled system* (2.1)*, $u \in C^{1,1}$ and satisfies $0 \le u \le 1$, and if also, for $y \in I$,*

$$1 - u(0, x, y) \sim v_\lambda(y) e^{\lambda x} \qquad (x \to \infty),$$

then, as $t \to \infty$,

$$u(t, x+ct, y) \to w(x, y),$$

where

$$w(x,y) := \mathbb{E}_{x,y} \exp\big[-Z_\lambda(\infty)\big].$$

This function w *satisfies the travelling-wave equation* (2.2) *and is, modulo translations, the unique monotone wave of speed* c *from* S *to* T.

Proof. We are guided by McKean [49]. So, we are supposing that u solves (2.1), that $0 \le u \le 1$ and that

$$1 - u(0, r, y) \sim v_\lambda(y) e^{\lambda r} \qquad (r \to \infty). \tag{6.9}$$

For (temporarily) fixed $\epsilon > 0$, we have for large r,

$$\exp\left\{-(1+\epsilon) v_\lambda(y) e^{\lambda r}\right\} \le u(0, r, y) \le \exp\left\{-(1-\epsilon) v_\lambda(y) e^{\lambda r}\right\}.$$

Now since $L(t) + ct \to \infty$ (a.s.), we shall (a.s.) have for large t,

$$\exp\big\{-(1+\epsilon) Z_\lambda(t)\big\} \le \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big) \le \exp\big\{-(1-\epsilon) Z_\lambda(t)\big\}.$$

Thus we have that

$$\begin{aligned}
\liminf_{t\to\infty} \exp\big\{-(1+\epsilon) Z_\lambda(t)\big\} &\le \liminf_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big) \\
&\le \limsup_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big) \\
&\le \limsup_{t\to\infty} \exp\big\{-(1-\epsilon) Z_\lambda(t)\big\}.
\end{aligned}$$

Since $Z_\lambda(\infty)$ exists almost surely (and clearly all the terms in the sequence of inequalities are bounded below by 0 and above by 1) this becomes

$$\begin{aligned}
\exp\big\{-(1+\epsilon) Z_\lambda(\infty)\big\} &\le \liminf_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big) \\
&\le \limsup_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big) \\
&\le \exp\big\{-(1-\epsilon) Z_\lambda(\infty)\big\}.
\end{aligned}$$

After taking expectations and using Fatou's Lemma, this implies that

$$\mathbb{E}_{x,y} \exp\big\{-(1+\epsilon) Z_\lambda(\infty)\big\} \le \mathbb{E}_{x,y} \liminf_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t) + ct, Y_k(t)\big)$$

$$
\begin{aligned}
&\le \liminf_{t\to\infty} \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} u\big(0, X_k(t)+ct, Y_k(t)\big)\\
&\le \limsup_{t\to\infty} \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} u\big(0, X_k(t)+ct, Y_k(t)\big)\\
&\le \mathbb{E}_{x,y} \limsup_{t\to\infty} \prod_{k=1}^{N(t)} u\big(0, X_k(t)+ct, Y_k(t)\big)\\
&\le \mathbb{E}_{x,y} \exp\big\{-(1-\epsilon)Z_\lambda(\infty)\big\}.
\end{aligned}
$$

Using the McKean representation (4.19) this yields that

$$
\begin{aligned}
\mathbb{E}_{x,y} \exp\big\{-(1+\epsilon)Z_\lambda(\infty)\big\} &\le \liminf_{t\to\infty} u(t, x+ct, y)\\
&\le \limsup_{t\to\infty} u(t, x+ct, y)\\
&\le \mathbb{E}_{x,y} \exp\big\{-(1-\epsilon)Z_\lambda(\infty)\big\}.
\end{aligned}
$$

On letting $\epsilon \downarrow 0$, we now obtain the desired result

$$
u(t, x+ct, y) \to w(x,y) = \mathbb{E}_{x,y} \exp\big\{-Z_\lambda(\infty)\big\}. \tag{6.10}
$$

Existence of a monotone travelling wave from S to T when $c > c(\theta)$. It is now intuitively obvious, and not that difficult to prove directly from the branching property, that the function $w(\cdot,\cdot)$ in (6.10) is a monotone travelling wave from S to T. Firstly it does in fact satisfy the travelling wave equation (2.2) because it can be written in the form

$$
\begin{aligned}
w(x,y) = \int_{t=0}^{\infty} \Big(\exp\Big(-\big(r(y)+\theta q(y)\big)t\Big)\Big) \ &\Big\{ r(y) w^2\big(x+(b(y)+c)t, y\big) \\
&+ \theta q(y) w\big(x+(b(y)+c)t, \hat{y}\big)\Big\} dt
\end{aligned} \tag{6.11}
$$

where $\{y,\hat{y}\} = \{1,2\}$. Integrating by parts gives

$$
w(x,y) = \frac{r(y)}{r(y)+\theta q(y)} w^2(x,y) + \frac{\theta q(y)}{r(y)+\theta q(y)} w(x,\hat{y}) + \frac{b(y)+c}{r(y)+\theta q(y)} w'(x,y)
$$

since the last term in the integration by parts is simply a multiple of the x-derivative of equation (6.11).

Then note that

$$
w(x,y) = \mathbb{E}_{0,y} \exp\big\{-e^{\lambda x} Z_\lambda(\infty)\big\},
$$

(from the definitions of $\mathbb{E}_{x,y}$ and Z), that $w(x,y) \to 0$ as $x \to -\infty$ (because $Z_\lambda > 0$ (a.s.) and $\lambda < 0$), and that

$$
1 - w(x,y) \sim v_\lambda(y) e^{\lambda x} \qquad (x \to \infty) \tag{6.12}
$$

because Z_λ converges $\mathcal{L}^1$ (by Theorem 6.4). Hence

$$\mathbb{E}_{x,y} Z_\lambda(\infty) = \mathbb{E}_{x,y} Z_\lambda(0) = e^{\lambda x} v_\lambda(y).$$

Hence our claim is proved.

Uniqueness modulo translation of the monotone travelling wave from S to T. Let $c > c(\theta)$, and let $\tilde{w}$ be a monotone travelling wave from S to T. We know from differential-equation theory that

either a suitable translate of $\tilde{w}$ satisfies (6.12), *or* a suitable translate of $\tilde{w}$ satisfies

$$1 - \tilde{w}(x,y) \sim v_\beta(y) e^{\beta x} \qquad (x \to \infty), \tag{6.13}$$

where β is the monotone eigenvalue of $K_{c,\theta}$ further from 0.

If $\tilde{w}$ satisfies (2.2) and (6.12), then $u(t,x,y) := \tilde{w}(x-ct,y)$ satisfies (2.1) and (6.9), so that from (6.10), we must have $\tilde{w} = w$. If $\tilde{w}$ satisfied (6.13), then we would have

$$\tilde{w}(x,y) = \mathbb{E}_{x,y} \exp\{-Z_\beta(\infty)\} = 1,$$

because $Z_\beta(\infty) = 0$ (a.s.) by Lemma 6.5; and $\tilde{w}$ would not go from S to T.

The proof of uniqueness is now complete, so we have proven Theorem 6.6. □

It is interesting to compare the above probabilistic proofs of existence and uniqueness modulo translation of travelling waves with the analytic proofs given in Chapter 5. We could, for example, use ODE results to obtain results on $\mathcal{L}^1$ convergence of our martingales.

The methods in Chapter 5 dealt with the $c = c(\theta)$ case; the probability theory for this case should be amenable to the techniques of Neveu [53] — stopping lines will correspond to simple conditions on the occupation times of the two states of the Markov chain.

6.8 The Doob h-transform associated with equation (2.5)

For definiteness, we work once more with the $\mathbb{P} = \mathbb{P}_{0,1}$ measure. Suppose that $0 < \theta \leq \frac{2}{\sqrt{\rho_1 \rho_2}}$, so that E_+ and E_- exist. Let E_+ have coordinates (α_1, α_2) in $\mathbb{R}^2$. Suppose that $E_+ \neq (1,1)$. Then

$$M(t) := \alpha_1^{-1} \alpha_1^{N_1(t)} \alpha_2^{N_2(t)},$$

where $N_i(t)$ is the number of particles of type i at time t, is a positive martingale of constant expectation 1. We may therefore define a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_\infty$ via the fact that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = M(t) \text{ on } \mathcal{F}_t.$$

The measure $\tilde{\mathbb{P}}$ is associated with the set-up $(a_1, a_2, \tilde{q}_1, \tilde{q}_2, \tilde{r}_1, \tilde{r}_2, \theta)$ defined at (2.5) in exactly the same way as $\mathbb{P}$ is associated with the original set-up.

It is important to realize that though $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on every $\mathcal{F}_t$, they are mutually singular on $\mathcal{F}_\infty$. This is because $M(\infty) = 0$ almost surely ($\mathbb{P}$); for if we let (T_n) be the sequence of stopping times at which $N_1(t)$ increases by 1, then $M(T_n) = \alpha_1 M(T_n-)$, so that $M(\infty) = \alpha_1 M(\infty)$.

Chapter 7

Weak solutions and probability

7.1 Introduction

Chapter 6 presented a probabilistic study of a PDE system using the probabilist's golden rule that Itô's formula leads to martingales (see, for example, Rogers & Williams [62]). But the uses of Itô's formula involved the 'formal generator' of the branching process in a way which might cause some unease to analysts. The system studied so far concerns a simple system of two coupled first-order PDEs, a generalization of which we consider in this chapter. The solutions in which one is most interested have discontinuities which persist for all time, and therefore need to be interpreted as *weak* solutions (see Chapter 3).

Section 7.3 settles existence and uniqueness for such weak solutions, identifying a canonical '*exact*' solution which is *everywhere* defined. The direct method used is guided by the theory of measure-valued diffusions, MVDs, (see, for example, Dawson [22] and Dynkin [25]). (We stress that no knowledge of MVD theory is assumed here.) The method is more effective than the method of characteristics, and has the advantage that it leads immediately to the McKean representation without recourse to Itô's formula.

Notational point. Henceforth we *never* use u_t to denote $\frac{\partial u}{\partial t}$, rather u_t denotes u at time t.

7.2 A generalized hyperbolic system

Let I be a finite set (with the discrete topology); and let B and R be functions of I with $R \geq 0$. Let Q be an $I \times I$ matrix with non-negative off-diagonal elements and zero row sums.

Let f be a Borel function on $\mathbb{R} \times I$ with $0 \leq f \leq 1$. Let u, written $(t, x, j) \mapsto u_t(x, j)$ and regarded as a column vector in j when multiplied by Q, be a Borel function on $[0, \infty) \times \mathbb{R} \times I$ with $0 \leq u \leq 1$. Suppose that u is a weak solution of

$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial x} + Qu + R(u^2 - u), \quad \text{“}u_0(x,j) = f(x,j)\text{”}. \tag{7.1}$$

In full, the first equation reads:

$$\frac{\partial}{\partial t}u_t(x,j) = B(j)\frac{\partial}{\partial x}u_t(x,j) + \sum_k Q(j,k)u_t(x,k) + R(j)\left[u_t(x,j)^2 - u_t(x,j)\right].$$

By the statement that u is a weak solution, we mean that for $t > 0$ and a test function $\varphi \in C_K^{1,1,0}([0,t]\times\mathbb{R}\times I)$ (that is a function of compact support, continuously differentiable in $[0,t]$ and in space) thought of as a row vector in j,

$$\int_0^t \int_x \sum_j \left\{\frac{\partial\varphi}{\partial s} - \frac{\partial\varphi}{\partial x}B + \varphi(Q-R) + \varphi R u\right\}_s (x,j)u_s(x,j)dxdt \tag{7.2}$$

$$= \int_x \sum_j \varphi_t(x,j)u_t(x,j)dx - \int_x \sum_j \varphi_0(x,j)f(x,j)dx.$$

7.3 Analytic statement of some results

Shortly we shall shortly reformulate these results probabilistically.

Introduce the unique one-parameter (Markov) semigroup $\{P_t^{-R} : t \geq 0\}$ acting on $C_b(\mathbb{R}\times I)$ (suffix b standing for 'bounded') such that if $h(x,j) = e^{i\theta x}g(j)$, where $\theta \in \mathbb{R}$ and g is a function (or column vector) on I, then

$$\left(P_t^{-R}h\right)(x,j) = e^{i\theta x}\left(\exp\{(i\theta B + Q - R)t\}g\right)(j). \tag{7.3}$$

In regard to the existence of $\{P_t^{-R} : t \geq 0\}$, see the discussion around equation (7.5) below. By the Riesz representation theorem, $\{P_t^{-R} : t \geq 0\}$ has a canonical extension to a semigroup on $\mathcal{B}_b(\mathbb{R}\times I)$, the space of bounded Borel functions on $\mathbb{R}\times I$.

Equation (7.2) implies that for each $t > 0$, u satisfies:

$$u_t = P_t^{-R}f + \int_0^t P_{t-s}^{-R}(Ru_s^2)ds, \tag{7.4}$$

for almost every x. This is proved as follows. Standard Fourier theory shows that for $\psi_0(\cdot,\cdot)$ in $C_K^\infty(\mathbb{R}\times I)$,

$$\psi_r(x,j)dx := \int dx_0 \sum_{j_0} \psi_0(x_0,j_0)P_r^{-R}(x_0,j_0;dx,j) \quad (0 \leq r \leq t)$$

defines $\psi_\cdot(\cdot,\cdot) \in C_K^{1,1,0}([0,t]\times\mathbb{R}\times I)$ with

$$\frac{\partial\psi}{\partial r} = -\frac{\partial\psi}{\partial x}B + \psi(Q-R).$$

Now take $\varphi_s(x,j) = \psi_{t-s}(x,j)$ in equation (7.2).

But now define

$$\begin{aligned} u_t^{(1)} &:= P_t^{-R} f, \\ u_t^{(n+1)} &:= P_t^{-R} f + \int_0^t P_s^{-R} \left(R {u_{t-s}^{(n)}}^2 \right) ds, \quad (n \geq 1). \end{aligned}$$

Then it is almost immediate by the usual Picard/Gronwall argument that $u^* := \lim u^{(n)}$ exists *monotonically and uniformly on each* $[0,t] \times \mathbb{R} \times I$, and so gives *the exact* solution of equation (7.4): it is a solution in which there are no 'exceptional sets'; and if v is another exact solution to equation (7.4) with $0 \leq v \leq 1$, then v is equal to u^* *everywhere.* Moreover, u^* is a weak solution of equation (7.1).

7.4 Probabilistic interpretations/proofs

Let $\{\eta_t : t \geq 0\}$ be a Markov chain on I with Q-matrix Q. Define

$$\xi_t := \xi_0 + \int_0^t B(\eta_s) ds,$$

and set

$$\left(P_t^{-R} h\right)(x,j) := \mathbb{E}^{x,j}\, h(\xi_t, \eta_t) \exp\left\{ -\int_0^t R(\eta_s) ds \right\},$$

where $\mathbb{E}^{x,j}$ is the measure corresponding to starting position $(\xi_0, \eta_0) = (x,j)$. Let us check that this agrees with the semigroup of equation (7.3). Fix $\theta \in \mathbb{R}$. For a vector g on I, define

$$Z_t(g) := \exp\left(-\int_0^t R(\eta_s) ds \right) e^{i\theta\xi_t} g(\eta)_t. \tag{7.5}$$

If we write $\doteq$ to signify equality modulo differentials of local martingales, then (see Rogers & Williams [62])

$$d\{g(\eta_t)\} \doteq (Qg)(\eta_t) dt,$$

and

$$dZ_t(g) \doteq \exp\left(-\int_0^t R(\eta_s) ds \right) e^{i\theta\xi_t} \Big([-R + i\theta B + Q] g \Big)(\eta_t) dt. \tag{7.6}$$

In fact, the difference between the integrals of the two sides of equation (7.6) is bounded on each $[0,t]$, and so is not just a local, but a true, martingale. Hence

$$\frac{d}{dt}(S_t g) = S_t \left(i\theta B + Q - R \right) g,$$

where S_t is the linear map on vectors on I defined by

$$S_t g := \mathbb{E}^{0,j} Z_t(g).$$

Since S_0 is the identity, equation (7.3) now follows.

Guided by McKean, we now construct a branching Markov process related to the hyperbolic equation (7.1). Time 0 sees the birth of one particle, labelled 1, which has 'type' $Y_1(0)$ in I and 'position' $X_1(0)$ in $\mathbb{R}$. At time $t \geq 0$, there are $N(t)$ particles which, when labelled in order of birth, have 'types' $Y_1(t), \dots, Y_{N(t)}(t)$ in I, and positions $X_1(t), \dots, X_{N(t)}(t)$ in $\mathbb{R}$. The type of each particle behaves (independently of previous history, of the behaviour of other particles currently alive, etc) as a Markov chain on I with Q-matrix Q. A particle of type j moves on $\mathbb{R}$ with constant speed $B(j)$, and gives birth to a new particle of its own type with rate $R(j)$, so that in small time h, independently of 'everything else', it gives birth with probability $R(j)h + o(h)$. Particles live for ever, once born. We write $\mathbb{P}^{x,j}$ and $\mathbb{E}^{x,j}$ for the probability and expectation corresponding to the situation when $X_1(0) = x$ and $Y_1(0) = j$.

With f as our 'initial value for u_0', let

$$\Pi(t) := \prod_1^{N(t)} f\Big(X_k(t), Y_k(t)\Big), \quad v_t(x,j) := \mathbb{E}^{x,j}\Pi(t).$$

We now utilise an obvious argument. Let T be the time of the first birth after time 0, so that T is the birth-time of particle 2. Because of the rôle of R as birth-rate function, we have

$$\mathbb{P}^{x,j}\left(T \in ds \,\Big|\, Y_1(r) : r \leq s\right) = R\big(Y_1(s)\big) \exp\left\{-\int_0^t R\big(Y_1(r)\big)dr\right\} ds.$$

From time T on, the family tree of particle 2 evolves independently of its complement in the family tree of particle 1. We therefore have

$$\mathbb{E}^{x,j}\left(\Pi(t) \,\Big|\, T = s;\, Y_1(r) : r \leq s\right) = v_{t-s}\big(X_1(s), Y_1(s)\big)^2 \quad (s \leq t).$$

Hence, for $s \leq t$,

$$\mathbb{E}^{x,j}\left(\Pi(t); T \in ds \,\Big|\, Y_1(r) : r \leq s\right)$$

$$= R\big(Y_1(s)\big) v_{t-s}\big(X_1(s), Y_1(s)\big)^2 \exp\left\{-\int_0^t R\big(Y_1(r)\big)dr\right\} ds,$$

so that

$$\begin{aligned} \mathbb{E}^{x,j}\left(\Pi(t); T \in ds\right) &= \mathbb{E}^{x,j} R(\eta_s) v_{t-s}(\xi_s, \eta_s)^2 \exp\left\{-\int_0^t R(\eta_r)dr\right\} ds \\ &= \left\{P_s^{-R}\left(R v_{t-s}^2\right)\right\}(x,j)ds. \end{aligned}$$

Since

$$\mathbb{E}^{x,j}\left(\Pi(t); T > t\right) = \left(P_t^{-R} F\right)(x,j),$$

we see that v satisfies equation (7.4) exactly. Hence $v = u^*$ *everywhere*, and we have the McKean representation

$$u_t^*(x,j) = \mathbb{E}^{x,j}\Pi(t).$$

7.5 Convergence to travelling waves: the easy case

Suppose that

$$f(x,j) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

so that

$$u_t(x,j) = \mathbb{P}^{x,j}(L(t) > 0) = \mathbb{P}^{0,j}(L(t) + x > 0)$$

where

$$L(t) := \min\{X_k(t) : k \le N(t)\}.$$

Condition (†)

We consider the very special situation in which

$$Q \text{ is irreducible and there is a state } j_0 \in I \text{ with}$$

$$B(j_0) < B(j) \text{ for } j \ne j_0 \text{ and for which } R(j_0) > -Q(j_0, j_0).$$

We will refer to these conditions as (†).

When (†) holds there will almost surely exist at some time a particle of type j_0 which has an infinite '*line of descent*' consisting entirely of particles of type j_0. Thus there will be a *random* interval $[\sigma, \infty)$, which we choose to be maximal, such that for some random constant A,

$$L(t) - B(j_0)t = A \quad \text{for} \quad t \in [\sigma, \infty).$$

Then

$$\begin{aligned} u_t(x - B(j_0)t, j) &= \mathbb{P}^{0,j}(x + L(t) - B(j_0)t > 0) \\ \to w(x,j) &= \mathbb{P}^{0,j}(A > -x), \end{aligned}$$

and $w(x + tB(j_0), j)$ is a travelling-wave solution of equation (7.1).

In this case,

$$u_t(x, j_0) = 1 \quad \text{if} \quad x > -tB(j_0),$$

$$u_t(x, j_0) = 1 - \mathbb{P}^{0,j_0}(L(t) = tB(j_0)) < 1 \quad \text{if} \quad x = -tB(j_0),$$

and the jump $\mathbb{P}^{0,j_0}(L(t) = tB(j_0))$ at the '*characteristic point*' $x = -tB(j_0)$ converges as $t \to \infty$ to

$$\mathbb{P}^{0,j_0}(\sigma = 0) = \frac{R(j_0) + Q(j_0, j_0)}{R(j_0)}.$$

For numerical and simulation studies of such a case, see Chapter 8 below.

7.6 The difficult cases

It is hoped to make the difficult cases when (†) fails to hold the subject of a paper giving direct proofs for this simple situation of results

$$u_{t+ct-a(t)}(x,j) \to w(x,j) \quad \text{where} \quad a(t) = o(t).$$

Indeed, $a(t)$ may behave like a multiple of $\log t$ or of $\log\log t$. Such results follow from deep results in existing literature. The classic paper on the '*logarithmic correction*' for the Fisher equation is that of Bramson [9].

Chapter 8

Numerical analysis

8.1 Introduction

We use both probabilistic simulation of the branching process and finite difference methods (the most effective being upwinding along characteristics) to study the initial value problem for a pair of coupled first order PDEs. These PDEs are those studied in Chapters 2–6 with θ set equal to 1 for clarity, which corresponds to the system studied in Chapter 7 when $I = \{1, 2\}$.

As in Chapter 3 it is convenient to change to moving coordinates (moving at a speed of $\frac{1}{2}(B(1) + B(2))$) and then re-scale time so that the coefficients of $\frac{\partial u}{\partial x}$ are 1 and -1. This is possible unless $B(1) = B(2)$ — this case reduces to a pair of first order ODEs and was dealt with in section 3.8. For the remainder of this chapter, we shall denote $u_t(x, 1)$ by $u(t, x)$ and $u_t(x, 2)$ by $v(t, x)$ and set $B(1) = 1, B(2) = -1$. Particularly interesting is Heaviside initial data, that is,

$$u(0, x) = v(0, x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

8.2 Finite difference methods

We present first the skeleton of the C program used to produce the numerical solution plotted in Figures 8-1 and 8-4. It implements a naive Euler method along the characteristics of the system, and a modification of the Euler method. The figures were produced using the modified method, but output of the two methods is practically indistinguishable.

```
/*  EULER METHODS
This is only part of a program.

Use of naive Euler methods for a simple hyperbolic system
   du/dt =  du/dx + f(u,v),  f(u,v) = q1 (v-u) + r1 u(u-1);
   dv/dt = -dv/dx + g(u,v),  g(u,v) = q2 (u-v) + r2 v(v-1);
```

```
with Heaviside initial data
     u(0,x) = v(0,x) =  1 for x >  0,
                        0 for x <= 0.

After the nth step, u[k] = uu[k+500] represents
     u(nh, 2kh)     if n is even;
     u(nh, (2k-1)h) if n is odd.

So, as it were, u[n,k] (that is, u[k] after n time steps)
corresponds to the pattern

t=2h             [2,1]         [2,0]         [2,1]
t= h                    [1,0]         [1,1]
t= 0             [0,1]         [0,0]         [0,1]
x-value          -2h     -h      0      h     2h

This is suited to integrating along the characteristics,
but we have to watch the parity.

nu denotes the next u-array, that is, u one time step later.
We consider the values -480 <= k <= 480; 0 <= n <= 960 (= bign),
with printouts every 160 (=gap) steps.                         */

#define bign 960
#define gap 160
double q1, q2, r1, r2, t;
int a, b, n;  /* a and b are x-values where there
                     are discontinuities */
double uu[1001], vv[1001], nuu[1001], nvv[1001];
double  *u= &uu[500]; double  *v= &vv[500];
double *nu=&nuu[500]; double *nv=&nvv[500];
int k, j, parity, method; double h;

void Solve(void); /*  calls up Display when appropriate */
void Euler1(void); void Euler2(void); double Trim(double z);

void Solve(void)
{  double temp;
   /* Initialize  */
   for(k=-485; k<= 0; k++) {u[k] = 0.0; v[k] = 0.0;}
   for(k=1; k<= 485; k++)  {u[k] = 1.0; v[k] = 1.0;}
```

```
   parity=0;

   for(n=1; n<=bign; n++)
   {  a = 1 - (n/2); b=(n+1)/2; parity = 1 - parity;
      for(k=a; k<= b; k++)
      {  j = k + 1 - parity;
         (method == 1)? Euler1(): Euler2();  }
      /* Stabilize */
      for (k=a; k<=b; k++)
      {  u[k] = Trim(nu[k]); v[k] = Trim(nv[k]);  }
      /* Display */
      if (n % gap == 0) {t = n * h; Display();}
   }
}
void Euler1(void) /* the most naive updating possible */
{  nu[k] = u[j] + h * f(u[j],v[j]);
   nv[k] = v[j-1] + h * g(u[j-1], v[j-1]);
}
void Euler2(void)  /* a refinement of the method */
{  double nu_temp, nv_temp;
   nu_temp = u[j] + h * f(u[j],v[j]);
   nv_temp = v[j-1] + h * g(u[j-1], v[j-1]);
   nu[k] = u[j] + 0.5*h*(f(u[j],v[j]) + f(nu_temp, nv_temp));
   nv[k] = v[j-1]+0.5*h*(g(u[j-1], v[j-1])+g(nu_temp, nv_temp));
}
double Trim(double z)
{  if (z>1.0) return 1.0
   else if (z<0.0) return 0.0
   else return z;
}
/* EOF */
```

Several finite difference methods were implemented on a rectangular lattice. These all proved to be less effective than the Euler method used along the characteristics (via the customized lattice, that is, using the characteristics to build the grid). The most effective of these schemes was the Lax-Wendroff scheme, as implemented in the following program. Further details of the methods tested and examples of their output are given in section 9.1 (and the rest of Chapter 9) — in this chapter we concentrate on the most successful work. Mitchell and Griffiths [51, Chapter 4] give a good discussion of the Lax-Wendroff scheme and hyperbolic equations in general; see also Strikwerda [68].

A solution plotted from this program is presented in Figure 8-3. It is fairly similar to the plots in Figures 8-1 and 8-2 for the same parameter values using the other methods investigated, but it suffers from typical Gibbs phenomena, and does not maintain the sharp discontinuities actually present in the true solution along each characteristic. However, for the parameter values given, these discontinuities decay exponentially to zero, and for longer times the solutions of all three methods agree very well (see section 9.1 for further discussion and graphs for this case). In cases where a discontinuity does not decay to zero (but instead to a finite size between 0 and 1, as in Figures 8-4 and 8-5), the Lax-Wendroff method is visibly worse (Figure 9-6) because the discontinuity is smeared out over several grid points. Away from the discontinuity agreement is good.

```
/* LAX-WENDROFF SCHEME
This is only part of a program.

Use of Lax-Wendroff scheme for a simple hyperbolic system
   du/dt =  du/dx + f(u,v),    f(u,v) = q1 (v-u) + r1 u(u-1);
   dv/dt = -dv/dx + g(u,v),    g(u,v) = q2 (u-v) + r2 v(v-1);
with Heaviside initial data
   u(0,x) = v(0,x) = 1 for x >  0,
                   = 0 for x <= 0.
After the nth step, u[k]=uu[k+500] represents u(n*lambda*h, k*h),
where lambda is the size of the time step divided by the size of
the space step (which should be below 1 for most of these
schemes).

nu denotes the next u-array, that is, u one time step later.

We consider the values -480 <= k <= 480; 0 <= n <= 480 (= bign),
with printouts every 80 (=gap) steps.                           */

#define gap  80   #define bign 480

double q1, q2, r1, r2, t;
int a, b, n; /* a and b control the region of the array in which
     calculation is performed, n current number of time steps */

double uu[1001], vv[1001], nuu[1001], nvv[1001];
double  *u =  &uu[500]; double *v  =  &vv[500];
double *nu = &nuu[500]; double *nv = &nvv[500];

int k;           /*    to be used as a counter variable       */
double h;
```

```
double lam;      /* lambda = time-step divided by space-step */

void Solve(void);       /* calls up Display when appropriate  */
double Trim(double z); /* ensures values stay in [0,1]        */
void LaxWen(void);     /* Lax-Wendroff method                 */

void Solve(void)
{
      /* Initialize */
  for(k=-485; k<= 0; k++) {u[k] = 0.0; v[k] = 0.0;}
  for(k=1; k<= 485; k++)  {u[k] = 1.0; v[k] = 1.0;}
  for(n=1; n<=L; n++) {
      a = 1 - n; b = n;
      for(k=a; k<= b; k++) {LaxWen();}
      }
       /* Stabilize */
   for(k=a; k<=b; k++)
      {u[k] = Trim(nu[k]); v[k] = Trim(nv[k]);}

      /* Display */
      if (n % gap == 0 ) {t = n * h * lam;Display();}
  }
}
void LaxWen(void)
{ nu[k] = u[k] + 0.5 * lam * (u[k+1] - u[k-1])
  + 0.5*lam*lam*(u[k+1] + u[k-1] - 2*u[k]) + h*lam*f(u[k],v[k]);
  nv[k] = v[k] - 0.5 * lam * (v[k+1] - v[k-1])
  + 0.5*lam*lam*(v[k+1] + v[k-1] - 2*v[k]) + h*lam*g(u[k],v[k]);
}
double Trim(double z)
{ if (z>1.0) return 1.0
  else if (z<0.0) return 0.0;
  else return z;
}
/* EOF */
```

8.3 Probabilistic simulation

The key to the probabilistic simulation is a simple recursive function called **Life**. This function tracks the path of an individual particle, updating the record of the left-most position yet reached by any particle each time the record is broken, and storing in arrays the time, place and type of each birth. Then, upon completion of a particle's run (since we only run each particle up until a pre-specified maximum time), we check to see if we have any more particles to do, and run the `Life` function on them.

Each particle is dealt with in a series of segments (the function `DoSegment`). An exponential random variable is generated (by `Rexp`)— the length of time until *either* the particle splits into two *or* the particle changes type. The position of the particle is updated by simply adding the length of time multiplied by its speed in its type for the segment to the current position. If the length of the segment takes us past our maximum time, we have completed the life story for that particle, and move on to the next one (incrementing `c`, the number of our current particle). Then we determine which event it was that did actually occur — birth or mutation. For birth we call the function `Create`, which stores the time, position and type in the arrays `tt`, `xx` and `yy` respectively. We also increment the counter `n` to inform us there is one more particle to be dealt with later. We then do another segment. If we change type, then we flip the type variable `y` and do the next segment.

Once we reach a point where we have completed the life story of a particle and there are no more sets of birth information unused (i.e. `c` greater than `n`), we have finished the simulation run.

If one is interested in calculating solely, for example, the left-most particle position, much calculation can be saved. Since a particle can only travel at speeds 1 and -1 we have immediate bounds on its future position (and identical bounds on all its future descendants). Thus, if the maximum time we are running until is T and the current record for left-most position is X, we can discard any particle in the simulation (denoting its position and time by (t, x)) for which $x - (T - t) > X$; particles satisfying this inequality have no chance of changing the record. The fact that their descendants also cannot break the record means we can discard the parent and save even generating the descendants.

```
/*  SIMULATION OF BRANCHING PROCESS MODEL

This is only part of a program. It shows how to extract
information on the left-most particle from the simulation (into
the arrays leftu and leftv), and save some computation if this
is all we are interested in (by pruning away particles far away
from the left-most).

While in type 1 particle moves at speed b[1] (wlog set to 1),
mutates at rate q[1] and breeds at rate r[1] - in type 2
particle moves at speed b[2] (wlog set to -1), mutates at rate
q[2] and breeds at rate r[2] - q and r user inputs.

The program notes the position of the left-most particle in
each of NUMRUN (1000) simulations starting from 1 particle at
```

```
the origin (doing 1000 runs for that particle being type 0, and
1000 for type 1), observing each simulation at T (6) points,
each GAP (user input) units apart.                           */

#define NUMRUN 1000
#define MAXPART 1000000 /* limit on particle numbers,
                          program warns if exceeded */
#define T 6
double b[3],Lpos[T+1],tt[MAXPART+1],xx[MAXPART+1],x,GAP,
       leftu[T+1][NUMRUN],leftv[T+1][NUMRUN],q[3],r[3];
int y,c,yy[MAXPART+1],TYPE,k;

void OneRun(void);
void DoSegment(void);
void Life(void);
void Create(void);

int main()
{
  int i,j; b[1]=1.0; b[2]=-1.0;
/* Simulations starting from 1 particle of type 1 */
  TYPE=1;
  for(i=0; i<NUMRUN; ++i) {
    OneRun();
    for(j=0;j<=T;++j) {
      leftu[j][i]=Lpos[j];
    }
  }
/* Simulations starting from 1 particle of type 2 */
  TYPE=2;
  for(i=0; i<NUMRUN; ++i) {
    OneRun();
    for(j=0;j<=T;++j) {
      leftv[j][i]=Lpos[j];
    }
  }
}
void OneRun(void)
{
  int i;
```

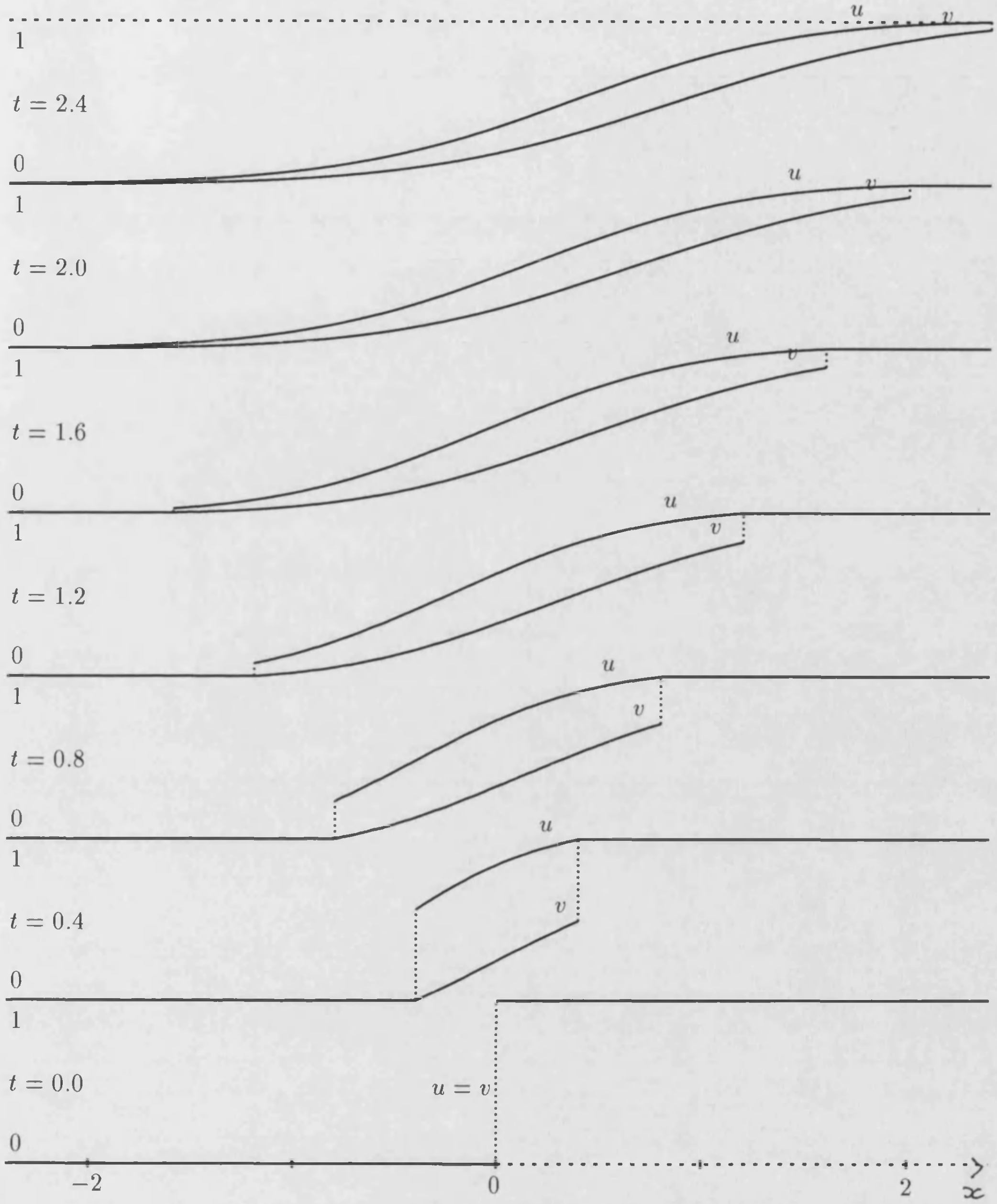
```
  n=0; c=0; t=0.0; x=0.0; Lpos[0]=x; tt[0]=t; xx[0]=x;
  yy[0]=TYPE;
  for(i=1; i <= T; ++i) {
    Lpos[i]=10000;
  }
  Life();
}
double RUnif() /* a different random number generator
                         could go here /*
{
  double drand48();
  return(drand48());
}
double Rexp(lam)
double lam;
{
  double log();
  return(-log(RUnif())/lam);
}
void DoSegment(void) /* update particles age and position */
{
  double e,Rexp();
  int i,j;

  e=Rexp(q[y]+r[y]); j=t/GAP; k=(t+e)/GAP;
  if (k > T)
    k=T;
  for(i = j + 1; i <= k; ++i) {
    if (Lpos[i] > x + b[y] * (i * GAP - t)) {
      Lpos[i]=x+b[y]*(i*GAP-t); /* this notes any new records
                                   set by this particle */
    }
  }
  x += e*b[y]; t += e;
}
void Life(void)
{
  double RUnif();

    /* initialise next particle */
  t = tt[c]; x = xx[c]; y = yy[c];
```

```
  while (t < T*GAP && (x+t-(T*GAP)) < Lpos[T]) {
    DoSegment();
    if (RUnif() < (r[y]/(q[y]+r[y])))
      Create();
    else
      y = 3 - y; /* This sends 1 to 2 and 2 to 1 */
  }
  c++;
  if (c >= MAXPART)
    printf("\nFilled up particle arrays\n");
  else if (c <= n)
    Life();
}
void Create(void) /* a new particle has been born,
                       store its details */
{
  n++;
  if (n < MAXPART)
     {tt[n]=t; xx[n]=x; yy[n]=y;}
}
/* EOF */
```

Plots of the solutions obtained by simulation are presented in Figures 8-2 and 8-5. These agree very well with those produced by the Euler method for corresponding parameters in Figures 8-1 and 8-4. When more simulation is done the probabilistic plots are smoother and agreement is even better. This simulation method could of course easily be extended to the n-type case.

8.4 Discussion of figures

Consider the situation in which there is just one particle at time 0, of type 2 and with position x. We know that $v(t,x)$ is the probability that all particles are to the right of 0 at time t. Since no particle can travel left at speed greater than 1, $v(t,x) = 1$ for $x > t$. However, if $x = t$, then $v(t,t)$ is the probability that up to time t our initial particle has no line of descent consisting only of particles of type 2: in other words, that every descendant of our initial particle spends some time before t moving right (in which case it can never get to 0 at time t). It is therefore clear that there is a positive jump $1 - v(t,t)$ in $v(t,\cdot)$ at time t, and that this jump may be calculated by regarding any particle of type 1 as 'dead' and ignoring it and its descendants. Precisely, the jump $1-v(t,t)$ is the probability that a continuous-time branching process starting from 1 particle, and with birth-rate r_2 and death-rate q_2 (per individual) survives until time

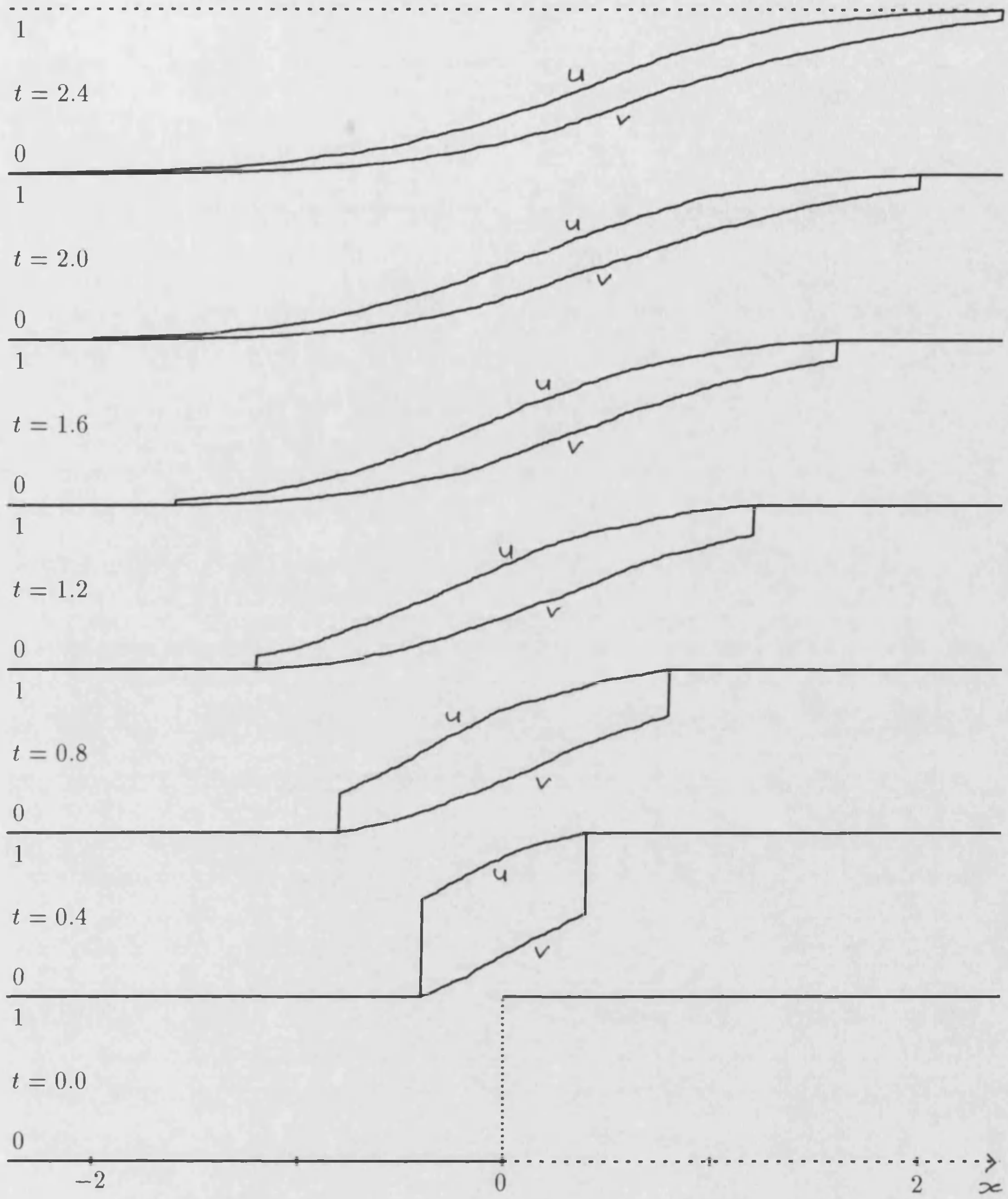


$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(0.400)2.400$

Method: Euler2 with $h = 0.0025$

Figure 8-1: Numerical solution calculated using modified Euler method along the characteristics

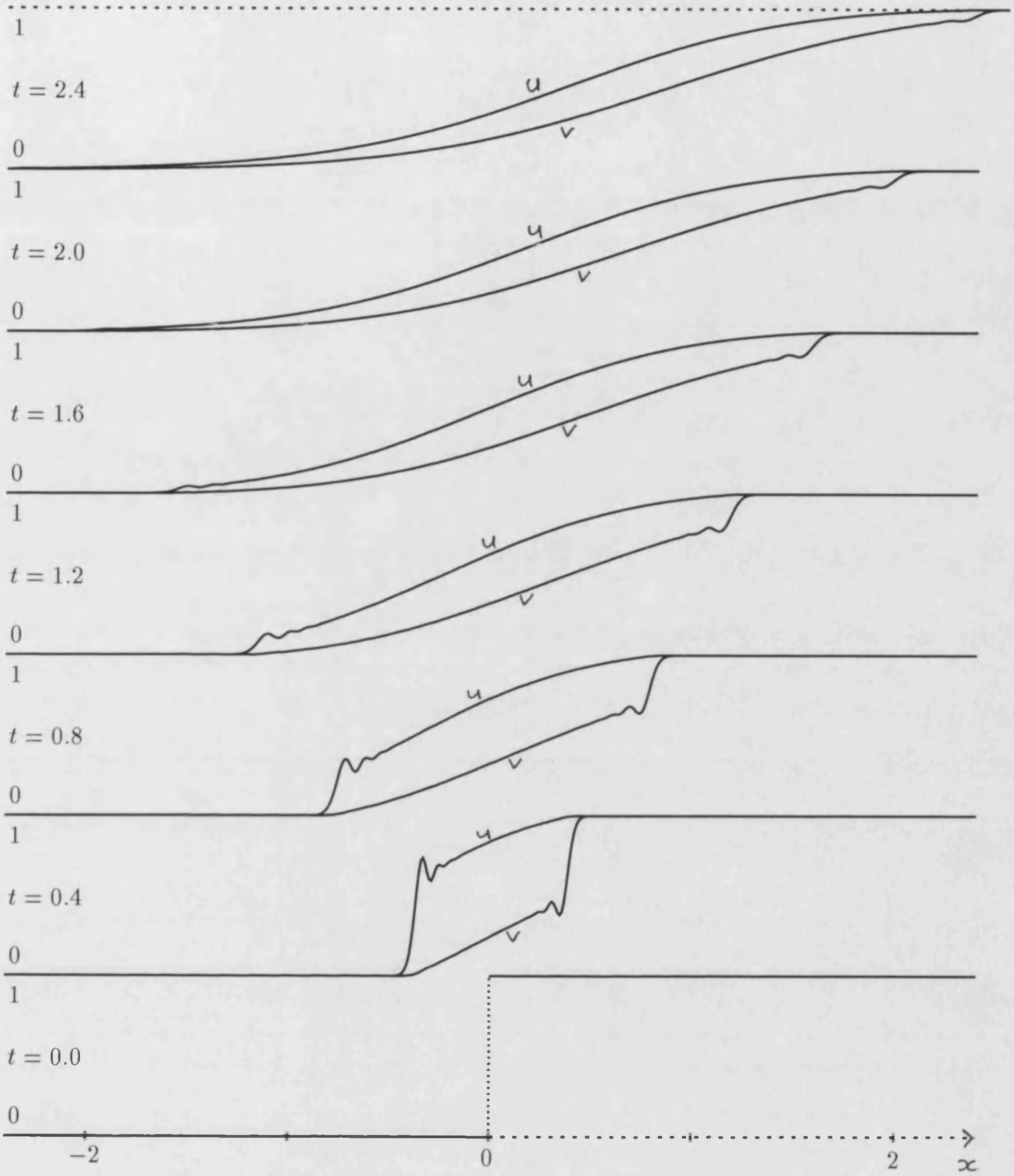


$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Distribution of left-most particle for $t = 0.0(0.400)2.400$

1000 Runs each from type 1, and from type 2, initial particle

Figure 8-2: Numerical solution calculated from probability simulation

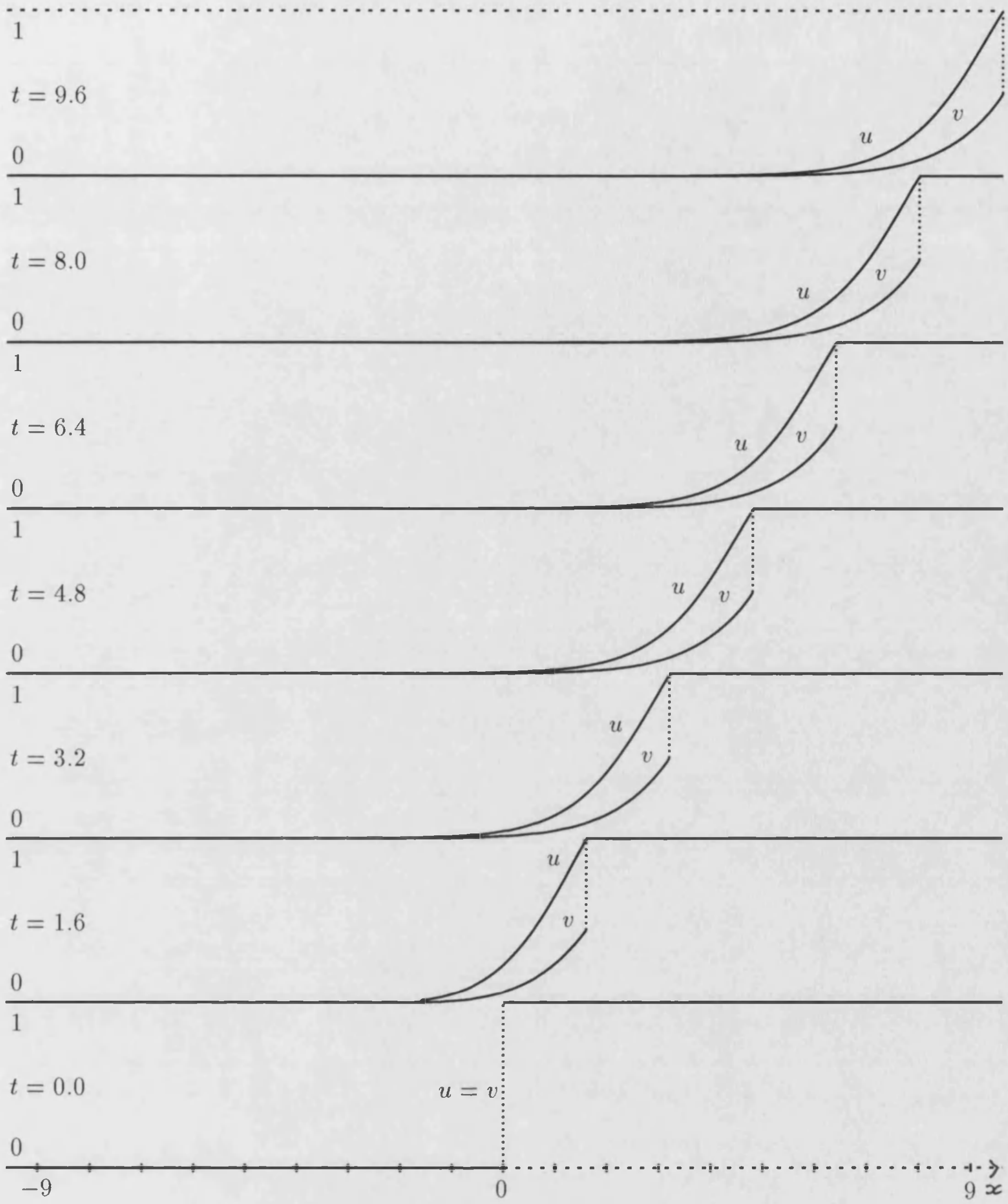


$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(0.400)2.400$

Method: Lax-Wendroff with $h = 0.010$, $\lambda = 0.5$

Figure 8-3: Numerical solution calculated using Lax-Wendroff method

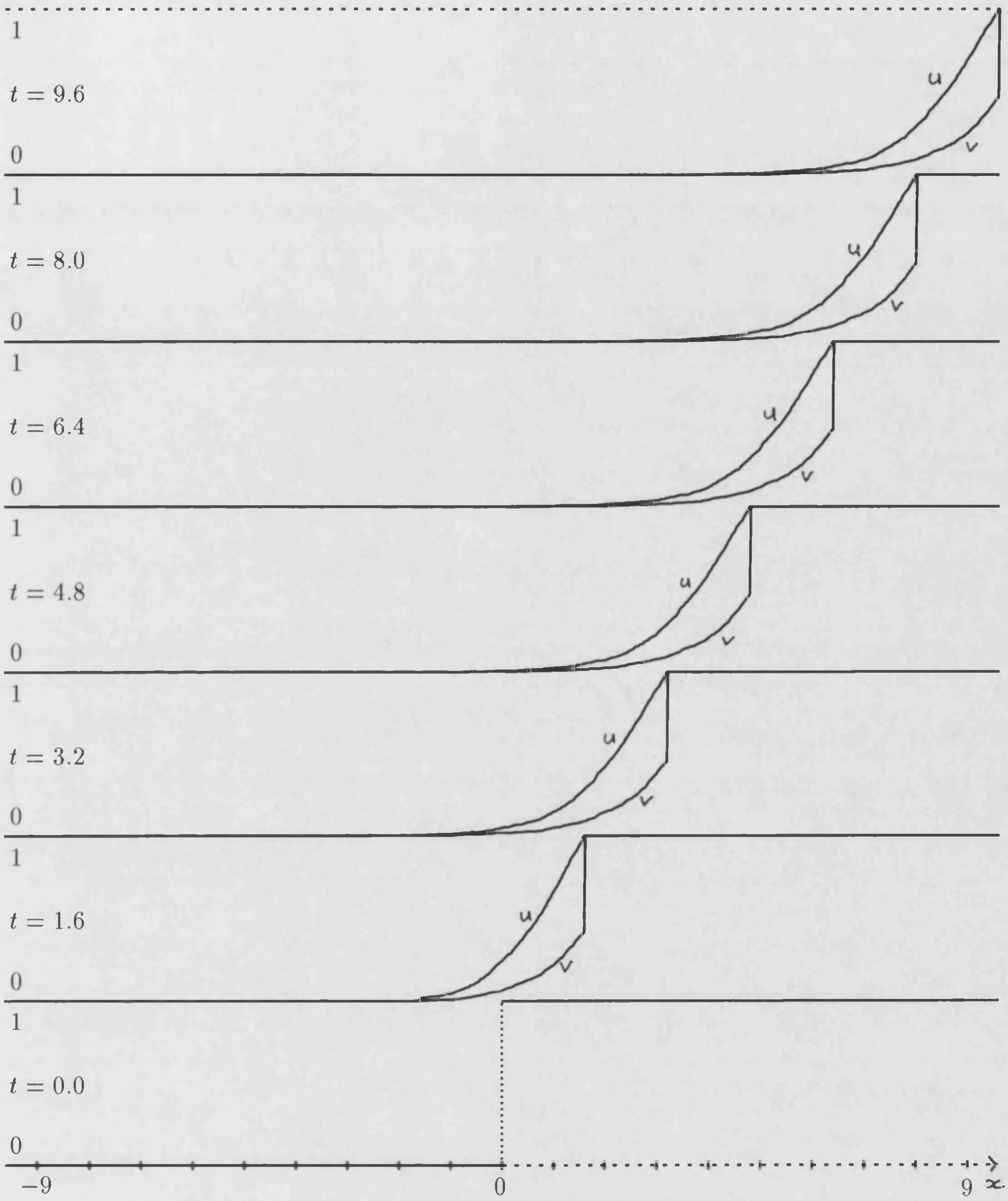


$q_1 = 2.00,\quad q_2 = 1.00,\quad r_1 = 1.00,\quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(1.600)9.600$

Method: Euler2 with $h = 0.0100$

Figure 8-4: Numerical solution calculated using modified Euler method along the characteristics



$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Distribution of left-most particle for $t = 0.0(1.600)9.600$

1000 Runs each from type 1, and from type 2, initial particle

Figure 8-5: Numerical solution calculated using probability simulation

t. If $q_2 > r_2$, the situation in Figures 8-1, 8-2 and 8-3, then this survival probability tends exponentially fast to 0. If $r_2 > q_2$, the situation in Figures 8-4 and 8-5, then $v(t,t) \to 1 - \pi$, where the extinction probability π satisfies

$$\pi = \frac{q_2}{q_2 + r_2} + \frac{r_2}{q_2 + r_2}\pi^2,$$

so that $\pi = q_2/r_2$.

Two other features of the pictures are worthy of comment. Firstly, the fast convergence to the travelling wave in Figures 8-4 and 8-5 illustrates the case discussed in section 7.5. Secondly, the almost linear nature of $v(t,\cdot)$ for small t, clearly apparent in Figures 8-1–8-3 may be explained as follows. Again, consider the situation in which there is just one particle at time 0, of type 2 and with position x. For small t, the dominant contribution to $v(t,x)$ will arise from cases where there is a random time S before t at which the particle changes type. Thus the particle moves left for a time S and right for a time $t - S$, ending up at $x - S + t - S$. Thus, for small t,

$$v(t,x) \approx \mathbb{P}\big(x - S + t - S > 0\big) = \mathbb{P}\Big\{S < \frac{1}{2}\big(x+t\big)\Big\} = 1 - e^{-\frac{1}{2}q_2(x+t)} \approx \frac{1}{2}q_2\big(x+t\big).$$

Further figures are presented and discussed in Chapter 9.

Chapter 9

Further numerical investigation

In this chapter we will give more details of our numerical work and show how it links with the theoretical development already given. In section 9.1 we compare the performance of the various integration schemes tested (for Heaviside initial data). In section 9.2 we use the phase plane to demonstrate how the numerical waves converge to travelling waves. In section 9.3 we experiment with some alternative initial data, cross-checking with the theory of Chapter 3 and in section 9.4 we show how a transformation extends the probabilistic interpretation outside the interval $[0, 1]$.

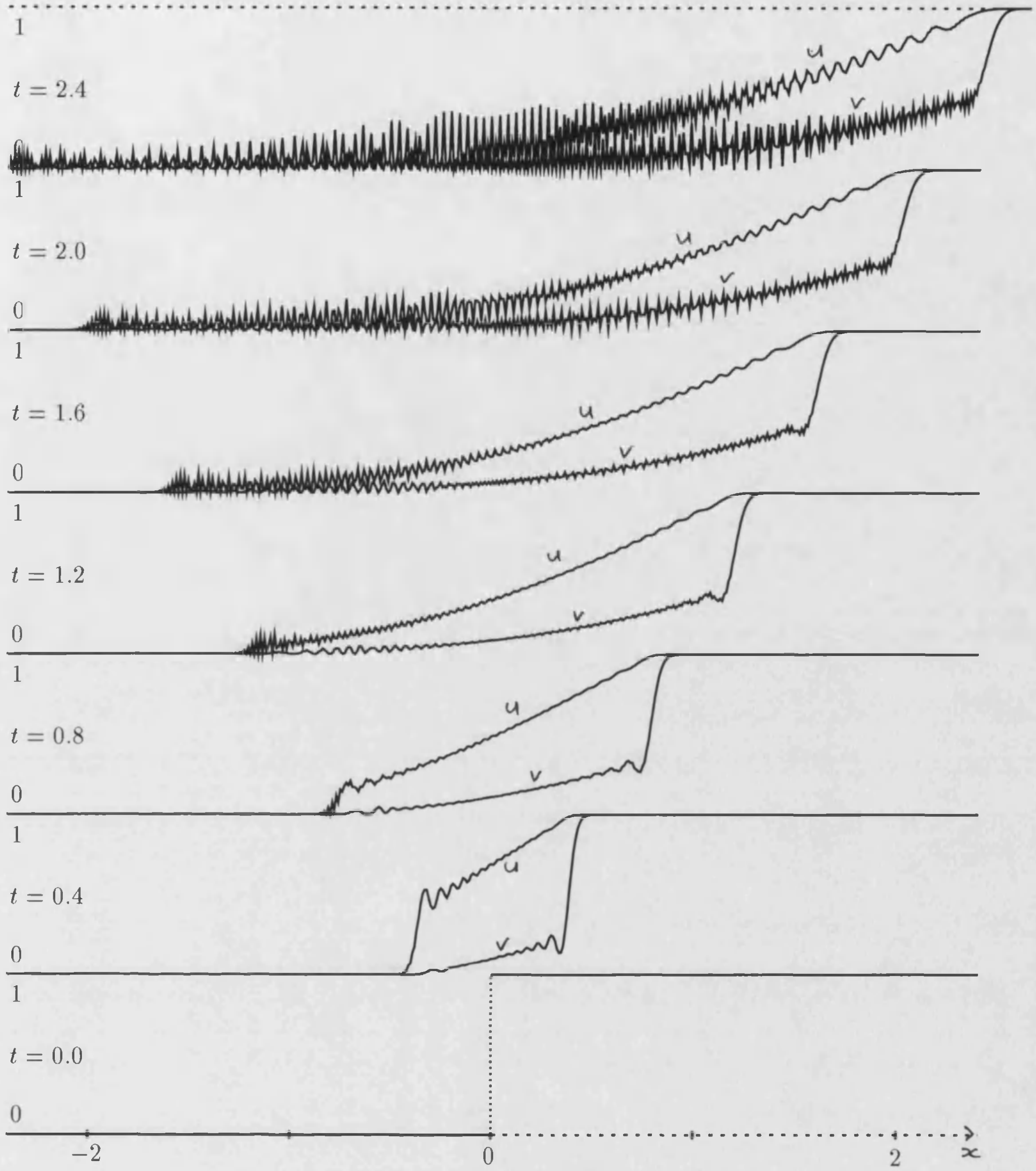
9.1 Comparison of finite difference methods

Two sets of methods were tested. On a standard rectangular grid we used the five given by Strikwerda [68, page 13] (forward-time, forward-space; forward-time, backward-space; forward-time, central-space; Lax-Friedrichs and leapfrog) and the Lax-Wendroff method. Since the problem is hyperbolic the other approach was to use the characteristics and build a grid using these — upwinding using simple Euler methods was done along the characteristics.

All the methods based on a rectangular grid had difficulties tracking the discontinuities in the solutions, the numerical solutions smeared out the sharp discontinuities. Gibbs phenomena were also very much in evidence, particularly in the simpler methods whose solutions broke down completely. The best of these methods was the Lax-Wendroff method, which was significantly better than the leapfrog method (the best of the rest, which is illustrated in Figure 9-1 for the same parameter values as Figures 9-4 and 9-5). The Lax-Wendroff method was the only one for which the magnitude of the Gibbs phenomena decreased over time, for the other methods the solutions deteriorated over successive time steps.

The methods using the characteristics gave much cleaner solutions, suffering from no Gibbs phenomena and, due to the grid using the characteristics, keeping accurate track of the discontinuities.

As can be seen in Figures 9-2 and 9-3, for parameter values for which both discontinuities decay to zero, the solutions obtained via Euler and Lax-Wendroff are practically indistinguish-

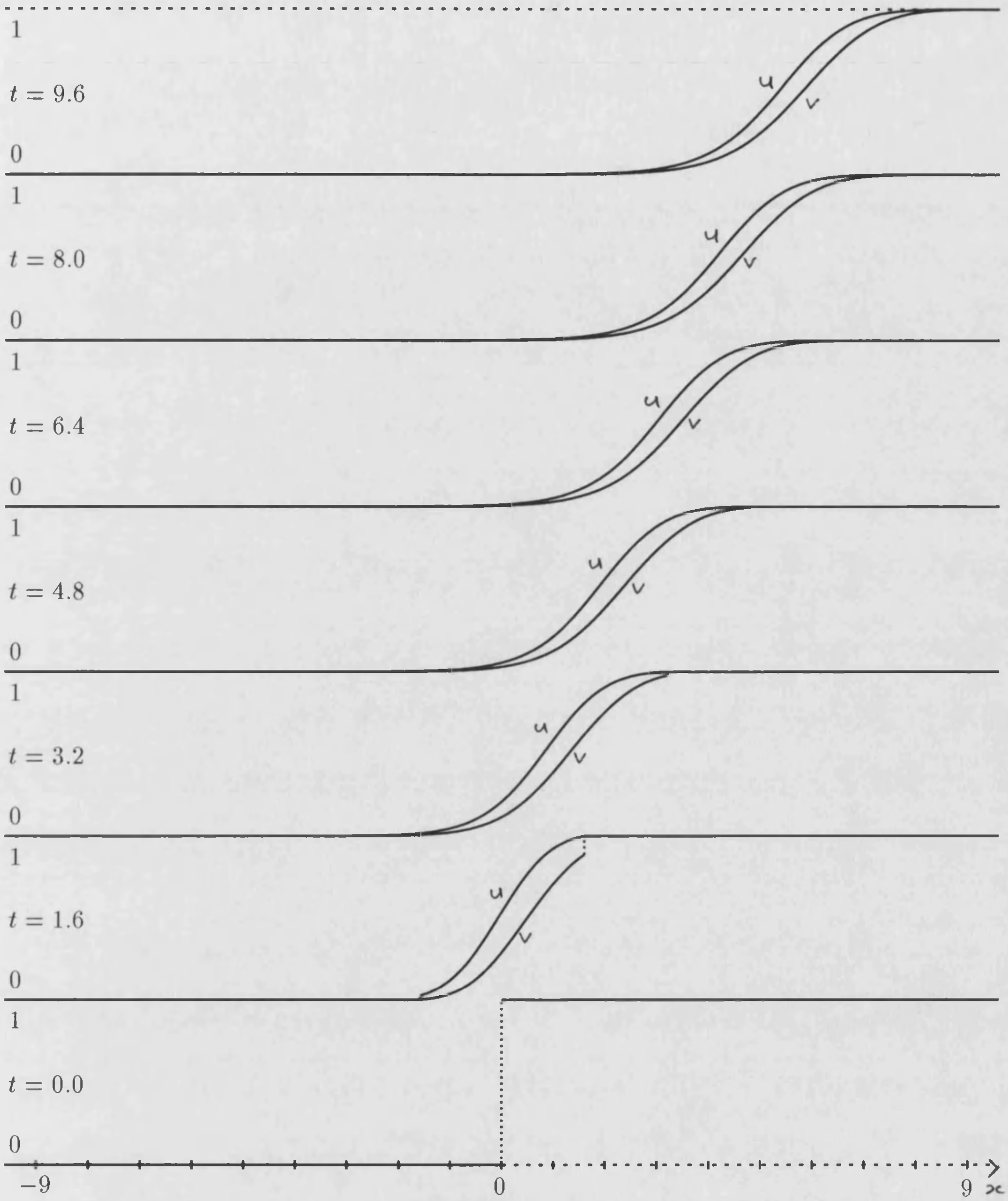


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(0.400)2.400$

Method: Leapfrog with $h = 0.010$, $\lambda = 0.500$

Figure 9-1: Numerical solution calculated using leapfrog method

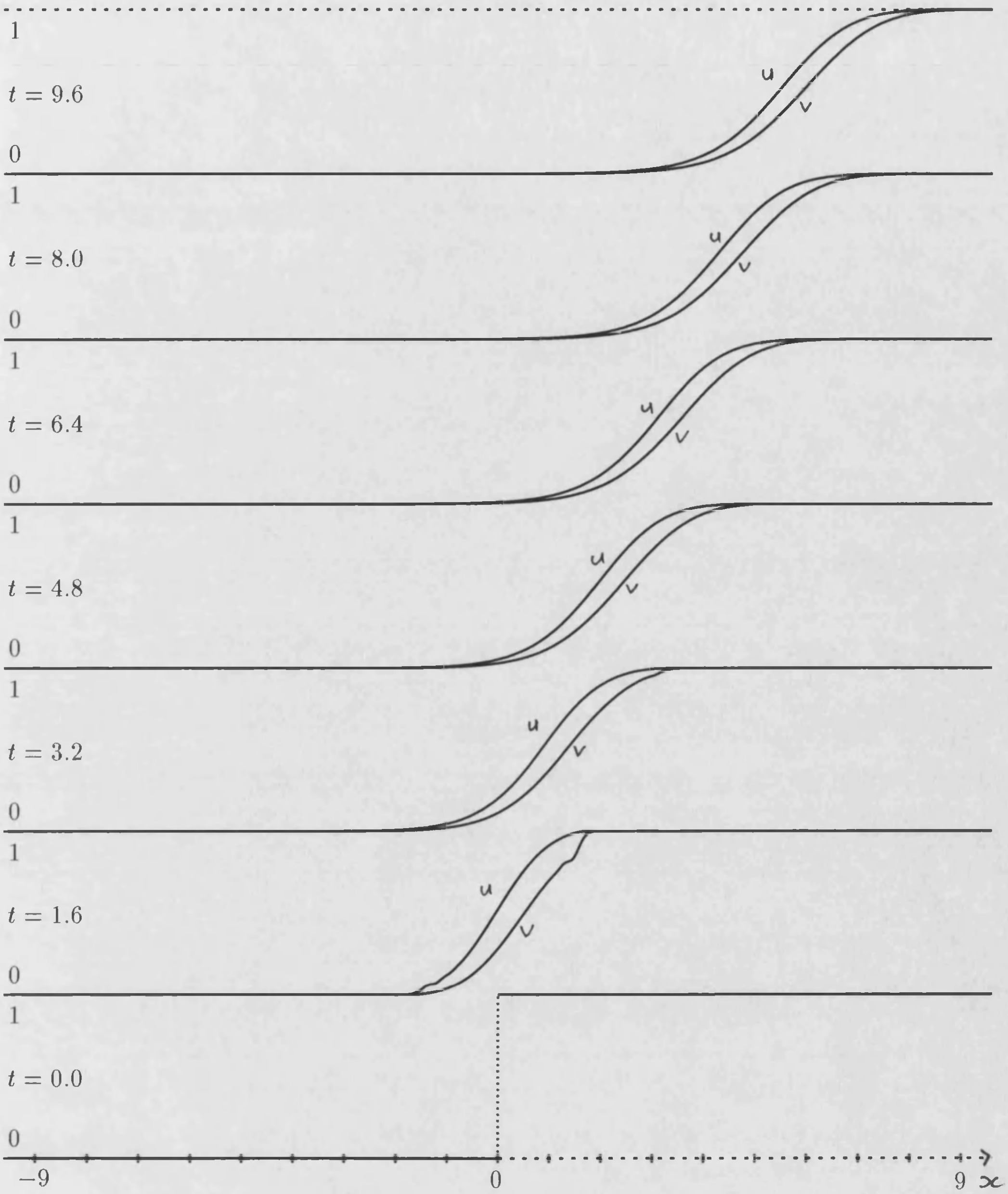


$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(1.600)9.600$

Method: Euler2 with $h = 0.0100$

Figure 9-2: Numerical solution calculated using modified Euler method along the characteristics



$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(1.600)9.600$

Method: Lax-Wendroff with $h = 0.040$, $\lambda = 0.5$

Figure 9-3: Numerical solution calculated using Lax-Wendroff scheme able for large times (and that obtained by probability simulation is also very similar, so is not presented). The initial difficulties the Lax-Wendroff scheme has with the discontinuity (as can seen from a comparison of Figures 8-1, 8-2 and 8-3 which are for the same parameter values but for a shorter time period) disappear as the discontinuity disappears.

On the other hand the diagrams in Figures 9-4 and 9-5 compare the short-time behaviour of the Euler (simulation agrees very well with the Euler method and is therefore not shown) and Lax-Wendroff approaches. Again observe the approximate linearity of the initial part of the solution (as discussed in section 8.4). The Gibbs phenomena in Figure 9-5 improve with time, but the plot still suffers badly from this failure to keep sharp discontinuities. Running the same parameter values for a longer time we obtain Figure 9-6 which is markedly inferior to Figures 8-4 and 8-5.

Henceforth we only present graphs produced from the Euler method along the characteristics.

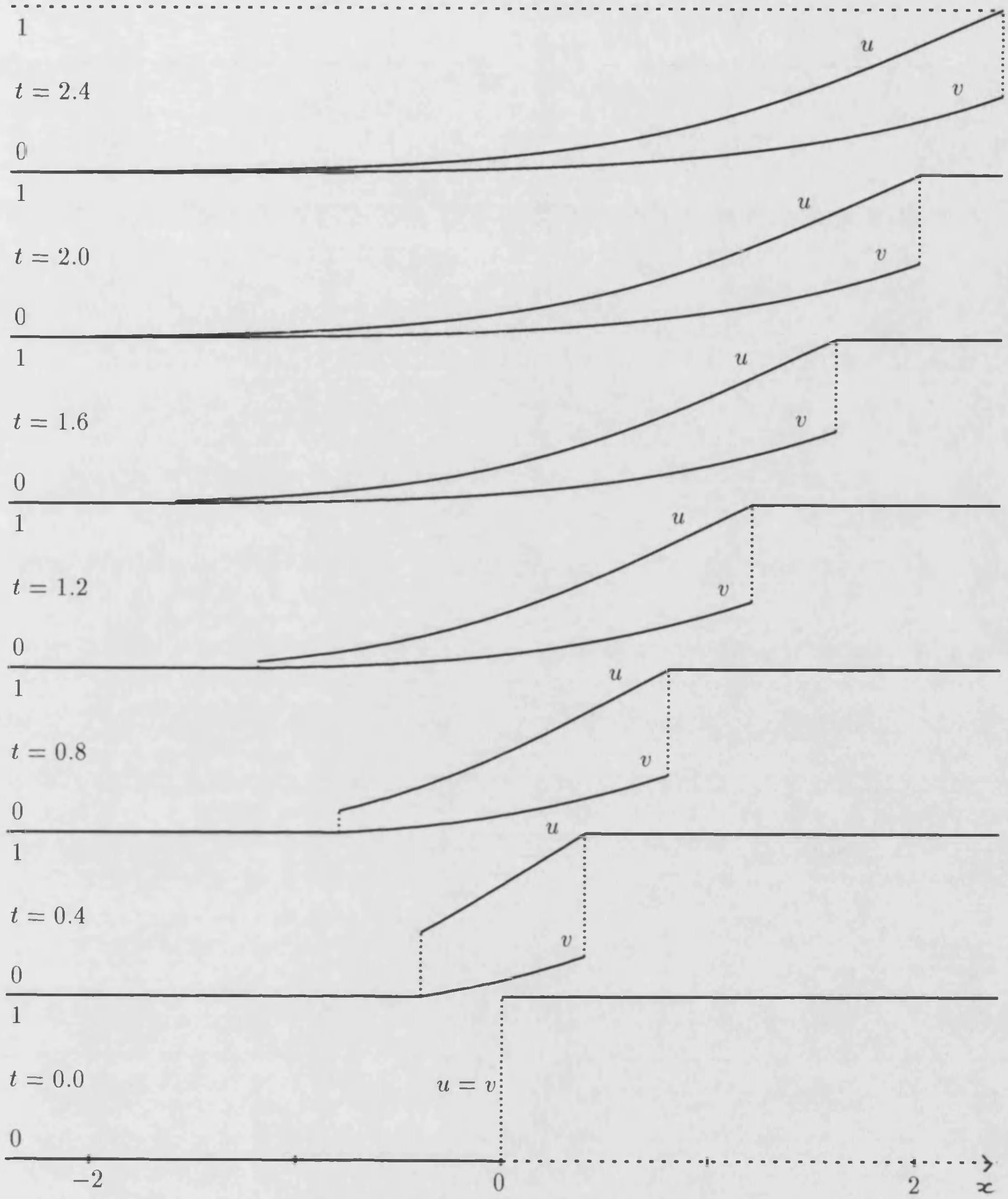
9.2 Convergence to travelling waves — viewed via the phase plane

It is clear from the plots of the numerical solutions (for example Figures 8-4 and 9-2) that convergence to a travelling-wave form occurs quite rapidly. An interesting way to observe this convergence is to plot the two components of the numerical solution against each other for a sequence of times — a phase plane type plot. The program implementing Euler methods on the characteristics was modified to do such plots, rather than plots of the two components through space. The grey lines on Figure 9-7 are the numerical solution observed at regular intervals (the darker the line, the later in time) and the black line is the parabolic segment of the nullcline nearest to the observed lines. As predicted by the preceding theory, when there is a persistent discontinuity (Figure 9-7 (b)) the travelling wave is simply the segment of the nullcline monotonically connecting $(0,0)$ and $(1,1)$ (see sections 5.6 and 7.5), when the discontinuity decays to zero (Figure 9-7 (a)) the travelling form is through the outside region (as predicted in section 5.6).

9.3 Alternative initial data

In this section we present some numerical results obtained for initial data (bounded between 0 and 1) other than the Heaviside initial data. Data bounded between $-K$ ($K > 0$) and 1 is discussed in section 9.4.

Figure 9-8 shows the numerical solution to the initial value problem with a short step function for initial data. Initially the two edges of the step function behave as the Heaviside initial data does, until they hit each other and collapse. This agrees with the comparison results in section 3.3 as this data can be bounded above by a pair of Heaviside step functions, one oriented leftwards and one rightwards. This data must obey the bounds obtained from each of

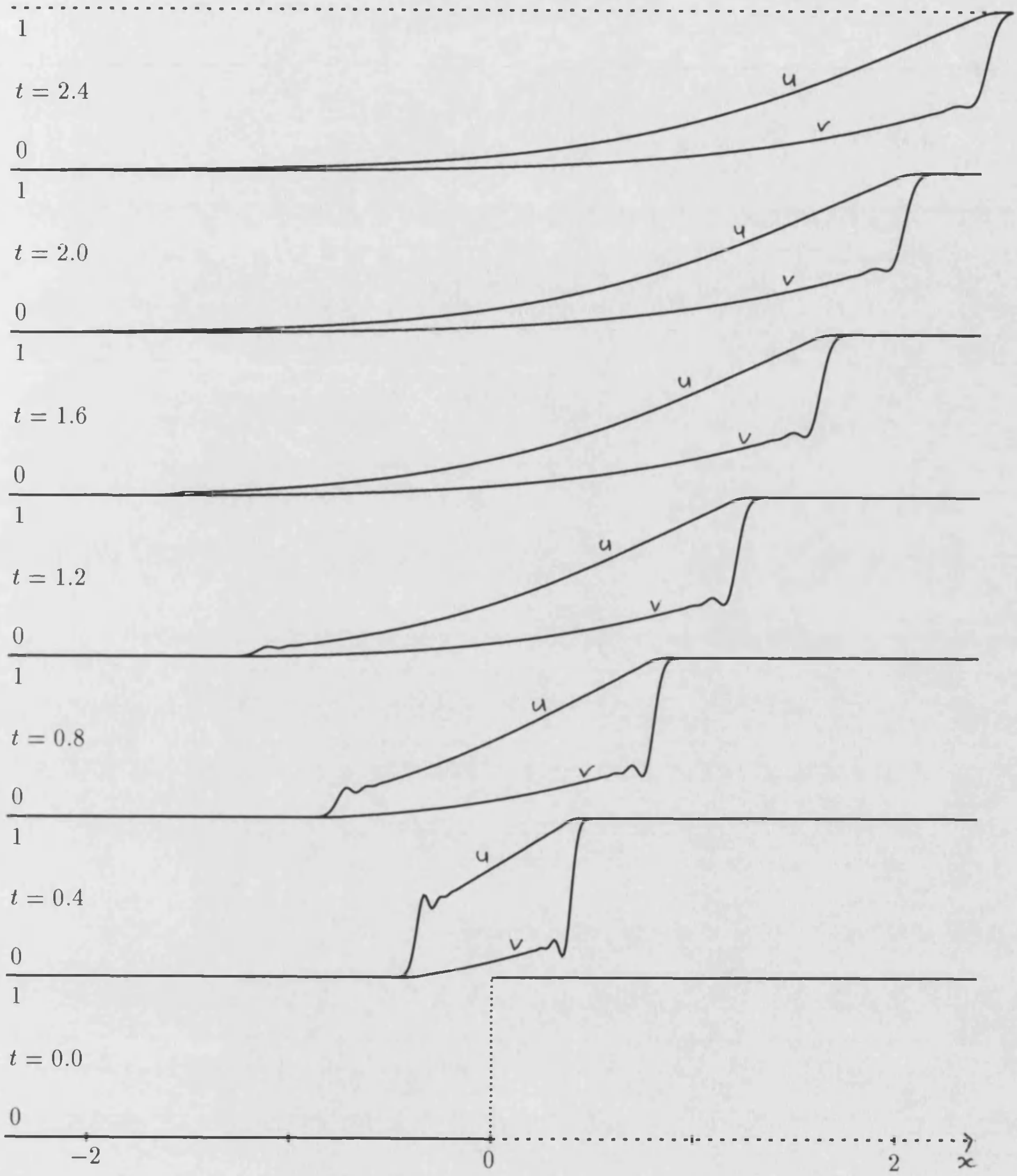


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(0.400)2.400$

Method: Euler2 with $h = 0.0025$

Figure 9-4: Numerical solution calculated using modified Euler scheme along the characteristics

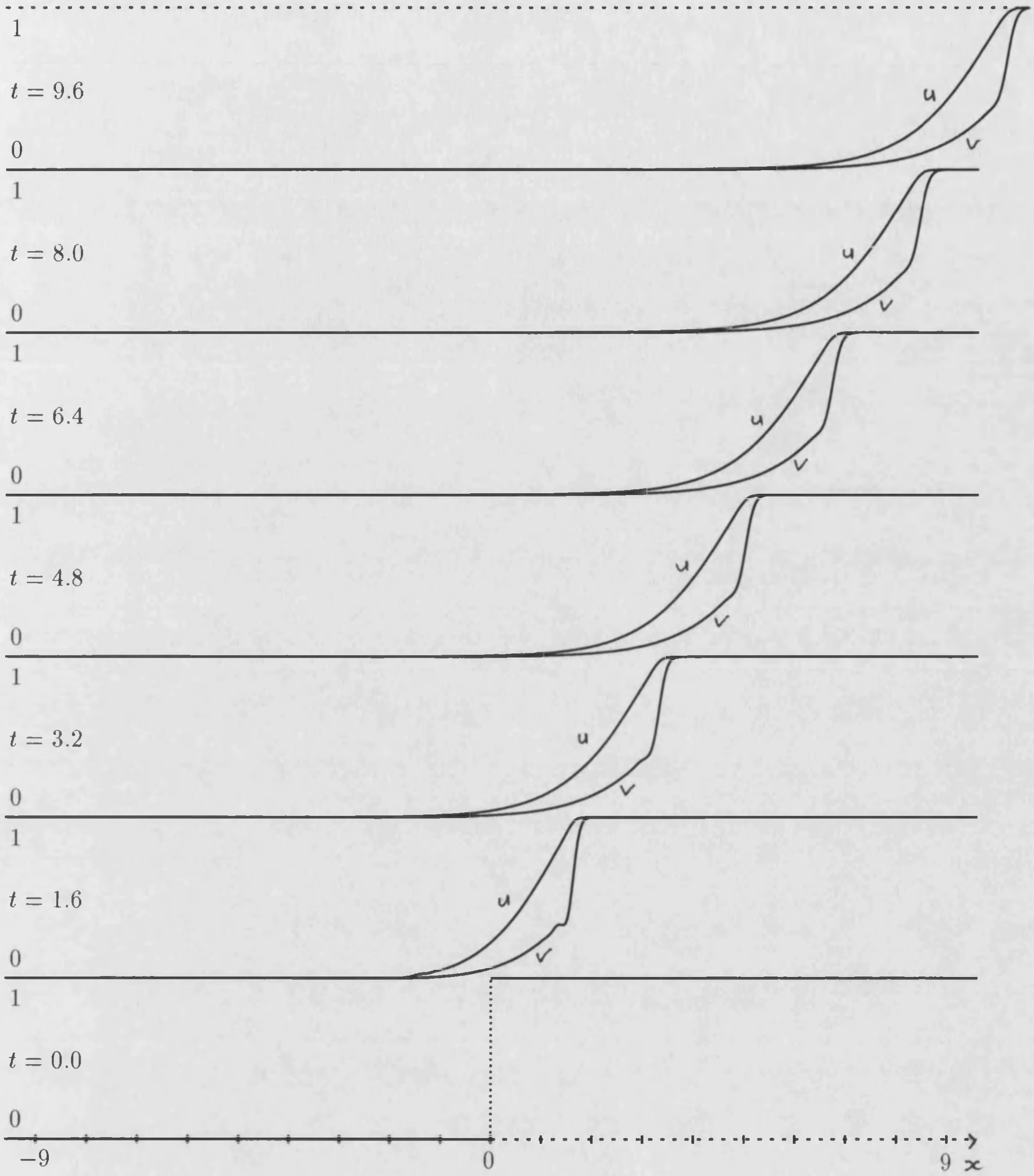


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(0.400)2.400$

Method: Lax-Wendroff with $h = 0.010$, $\lambda = 0.5$

Figure 9-5: Numerical solution calculated from Lax-Wendroff scheme

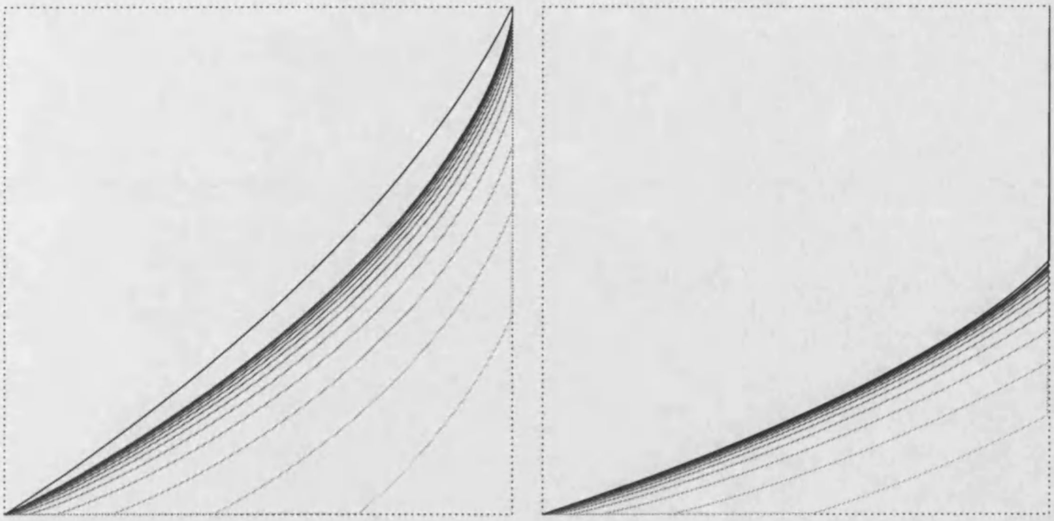


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(1.600)9.600$

Method: Lax-Wendroff with $h = 0.040$, $\lambda = 0.5$

Figure 9-6: Numerical solution calculated using Lax-Wendroff scheme



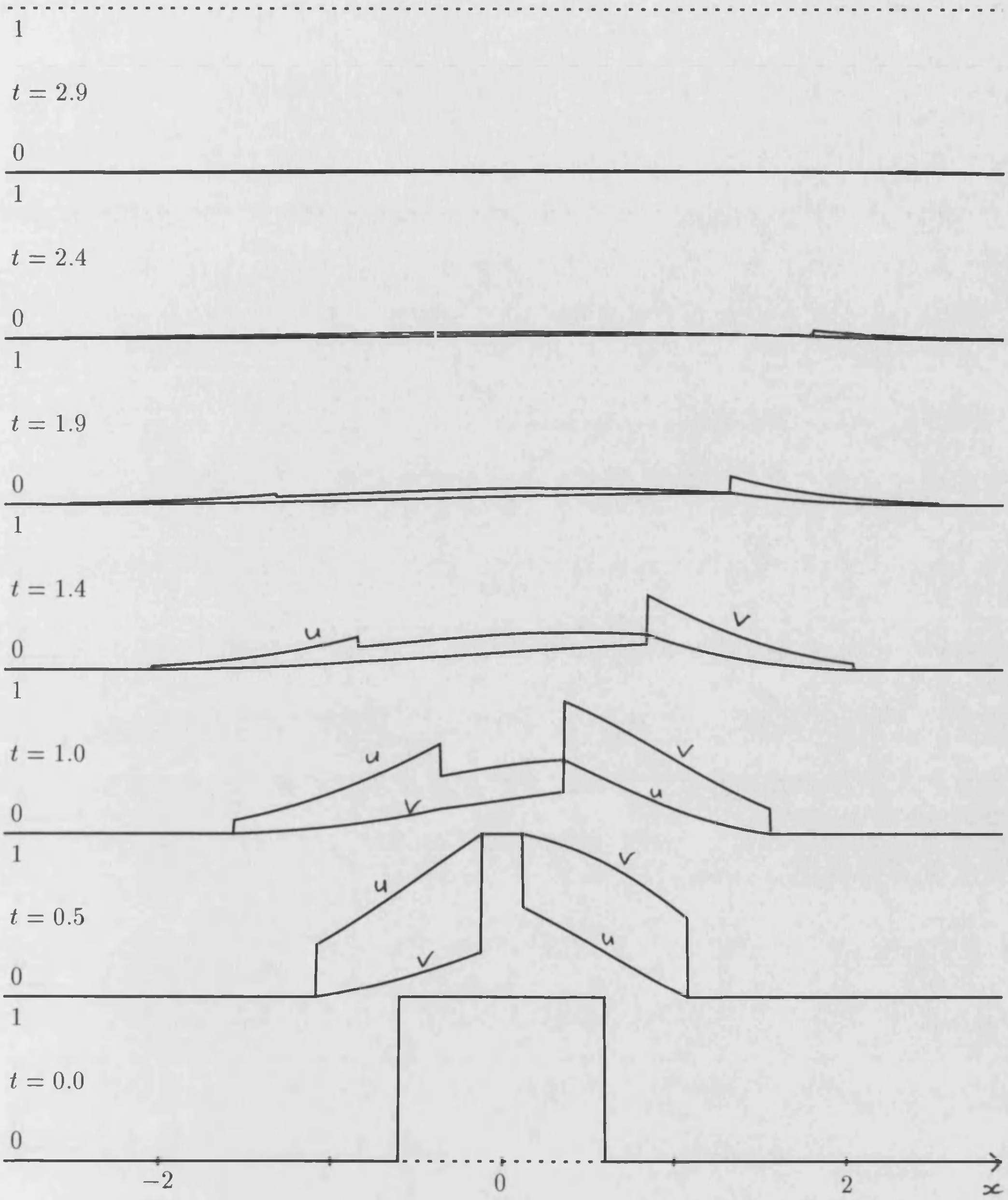
(a) $q_1 = 1, \quad q_2 = 2, \quad r_1 = 2, \quad r_2 = 1.$ (b) $q_1 = 2, \quad q_2 = 1, \quad r_1 = 1, \quad r_2 = 2.$

Figure 9-7: Plots of u against v for solutions to Heaviside initial value problem obtained using Euler 2 method with $h = 0.0035$. Lines plotted are at twelve time intervals of 0.28, running from time 0 to time 3.36.

these and so goes to zero rapidly everywhere. The probabilistic representation of the solution also explains the observed behaviour — the solution is non-zero only when the left-most and right-most particles are within a small, prescribed region, which in the long run will not occur. In the short term, the solution is identically 1 for a small region, corresponding to the fact that if the initial particle is near the centre of the region, then its descendants will still be inside it after a small time.

Figures 9-9 and 9-10 show the effect of different parameter values on a pair of Heaviside functions. The initial data is a Heaviside for each of u and v, but shifted relative to each other. The discontinuities in each component decay away, but in Figure 9-9 (parameter values which give rise to a persistent discontinuity for non-offset Heaviside data) the leading edge of one component sharpens up to approximate the discontinuous travelling wave form, while in Figure 9-10 the smooth form is approached.

Again these results can be seen to fit in with the comparison results — we can bound the initial data above and below by suitably placed Heaviside functions so we expect a solution travelling at the speed of non-offset Heaviside data. Probabilistically the initial data corresponds to a representation based on the position of the left-most particle of each type, with a shift of one type relative to the other. The figures show a flat section of the solution in the initial evolution of the solution, corresponding to a particle starting between the two critical

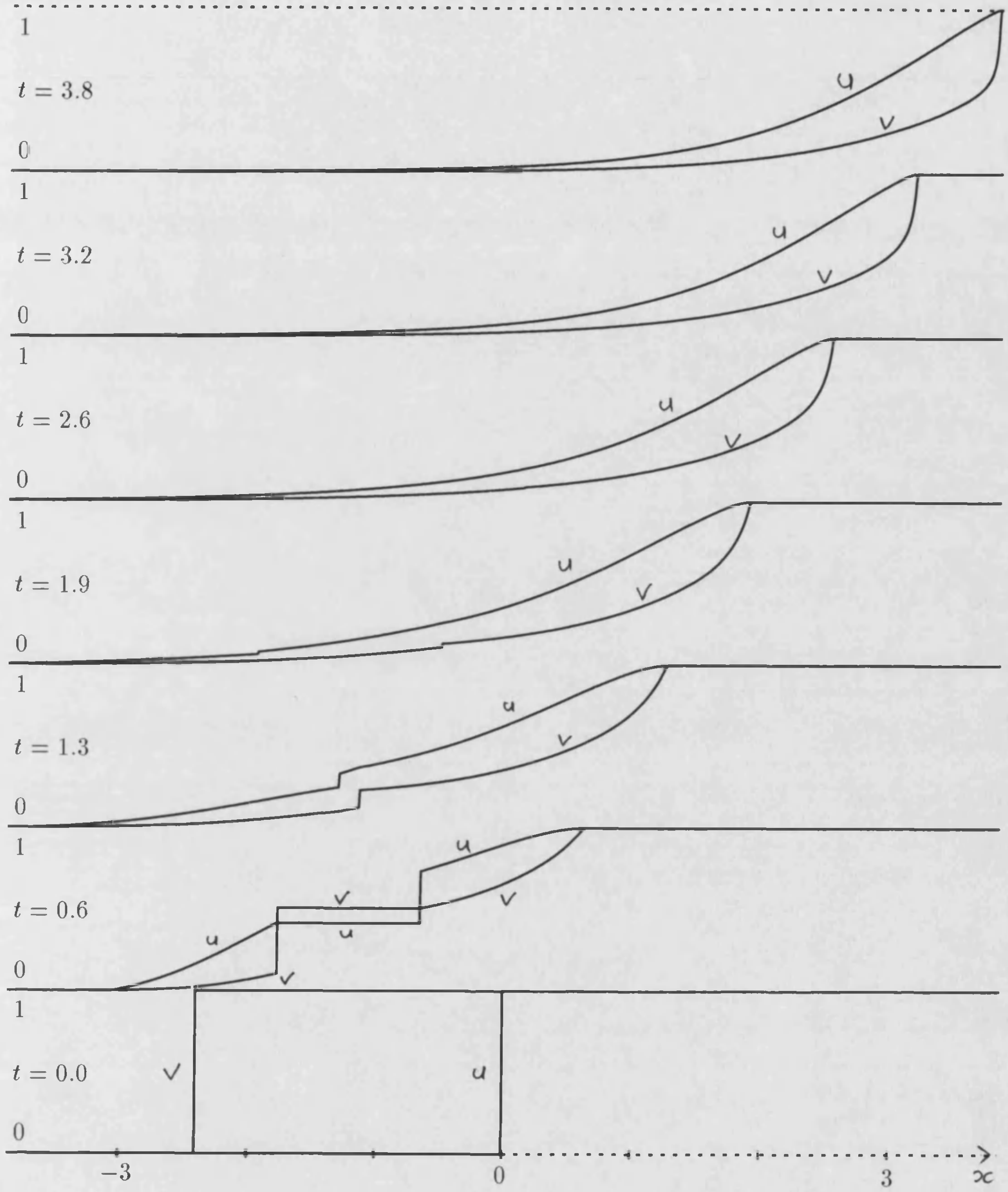


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(0.480)2.880$

IVP via Euler2 with $h = 0.003$

Figure 9-8: Initial step function analyzed by modified Euler method along characteristics

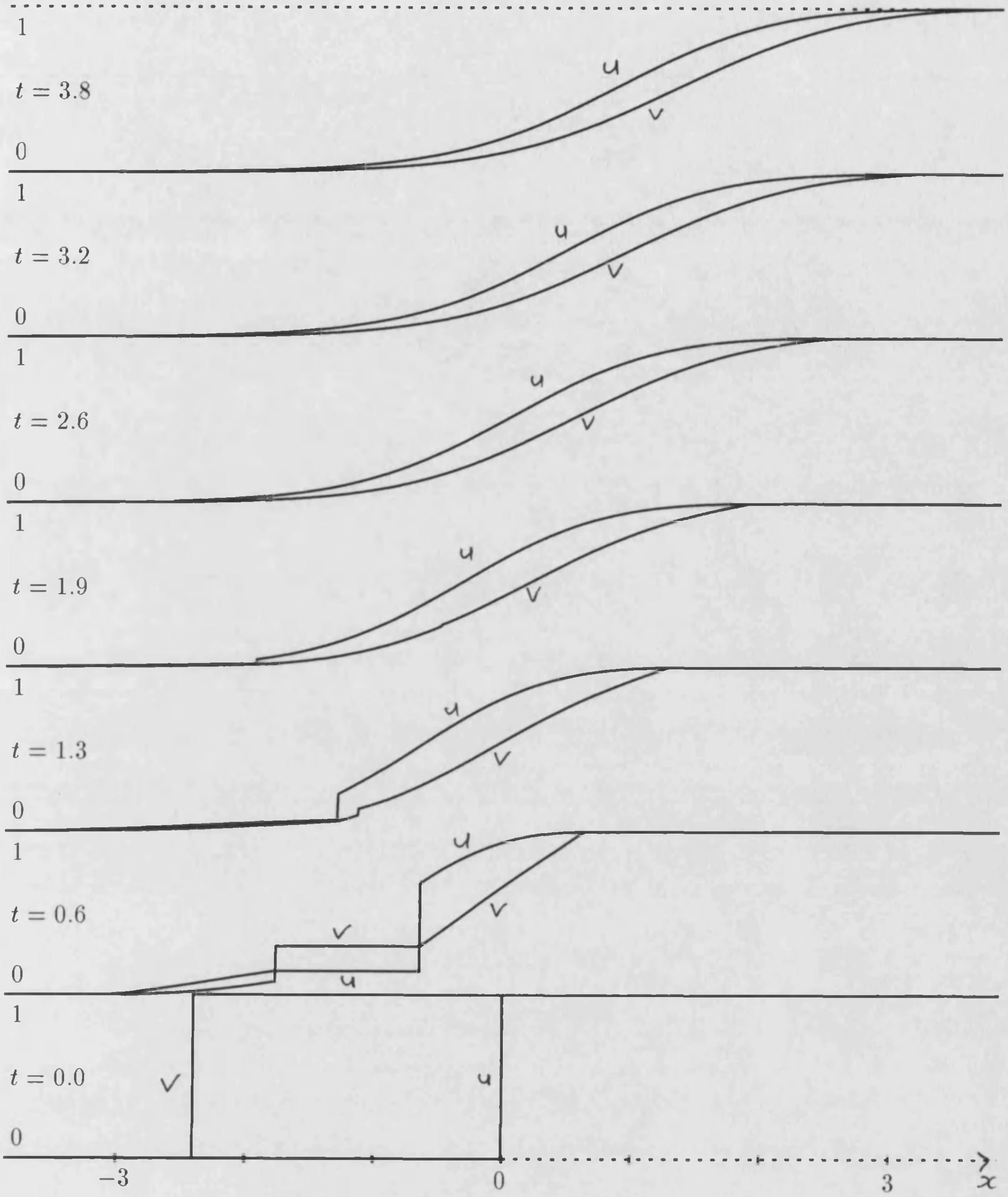


$q_1 = 2.00, \quad q_2 = 1.00, \quad r_1 = 1.00, \quad r_2 = 2.00.$

Graphs of u and v for $t = 0.0(0.640)3.840$

IVP via Euler2 with $h = 0.004$

Figure 9-9: Pair of offset Heaviside functions analyzed by modified Euler method along the characteristics



$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(0.640)3.840$

IVP via Euler2 with $h = 0.004$

Figure 9-10: Pair of offset Heaviside functions analyzed by modified Euler method along the characteristics edges — if it is of one type then it scores zero, if it is of the other type it scores 1. For short time the particle and all its descendants cannot escape this region, so the solution is zero unless they are all of the correct type. The chance of this happening is independent of the exact position within the region, so the solution is flat in this region (and large if the initial particle was of the right type, small if it was not).

9.4 Negative initial data and a transformation

The bounds on solutions for negative initial data in section 3.2 can be seen in terms of a McKean representation by use of a simple, scaling transformation. This transformation preserves the form of the partial differential equations, the only change coming in the non-linearity. This change can be interpreted as death being added to the model.

So, for this section let us consider the case in which the initial data is bounded between $-K$ and 1, for some $K > 0$ (then Lemma 3.6 guarantees existence and uniqueness of a solution to the PDE system (2.1) that respects these bounds for all time). Recall the notation of Chapter 2 in which (2.1) is as follows:

$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial x} + R(u^2 - u) + \theta Q u.$$

Define a new variable by

$$v = \frac{u+K}{1+K} = pu + q$$

where

$$p = \frac{1}{1+K} \qquad \text{and} \qquad q = \frac{K}{1+K}.$$

Notice that p and q are both strictly positive, and $p + q = 1$. Then, by substituting into equation (2.1), v satisfies the following equation:

$$\frac{1}{p}\frac{\partial v}{\partial t} = \frac{1}{p}B\frac{\partial v}{\partial x} + R\left(\left(\frac{v-q}{p}\right)^2 - \frac{v-q}{p}\right) + \theta Q\left(\frac{v-q}{p}\right). \tag{9.1}$$

Multiplying through equation (9.1) by p and noting that $Q(v - q) = Qv$ since Q has zero row sums yields

$$\frac{\partial v}{\partial t} = B\frac{\partial v}{\partial x} + R\left(\left(\frac{v^2 - 2qv + q^2}{p}\right) - (v - q)\right) + \theta Q v.$$

Using the definition of p and q this can be rewritten as:

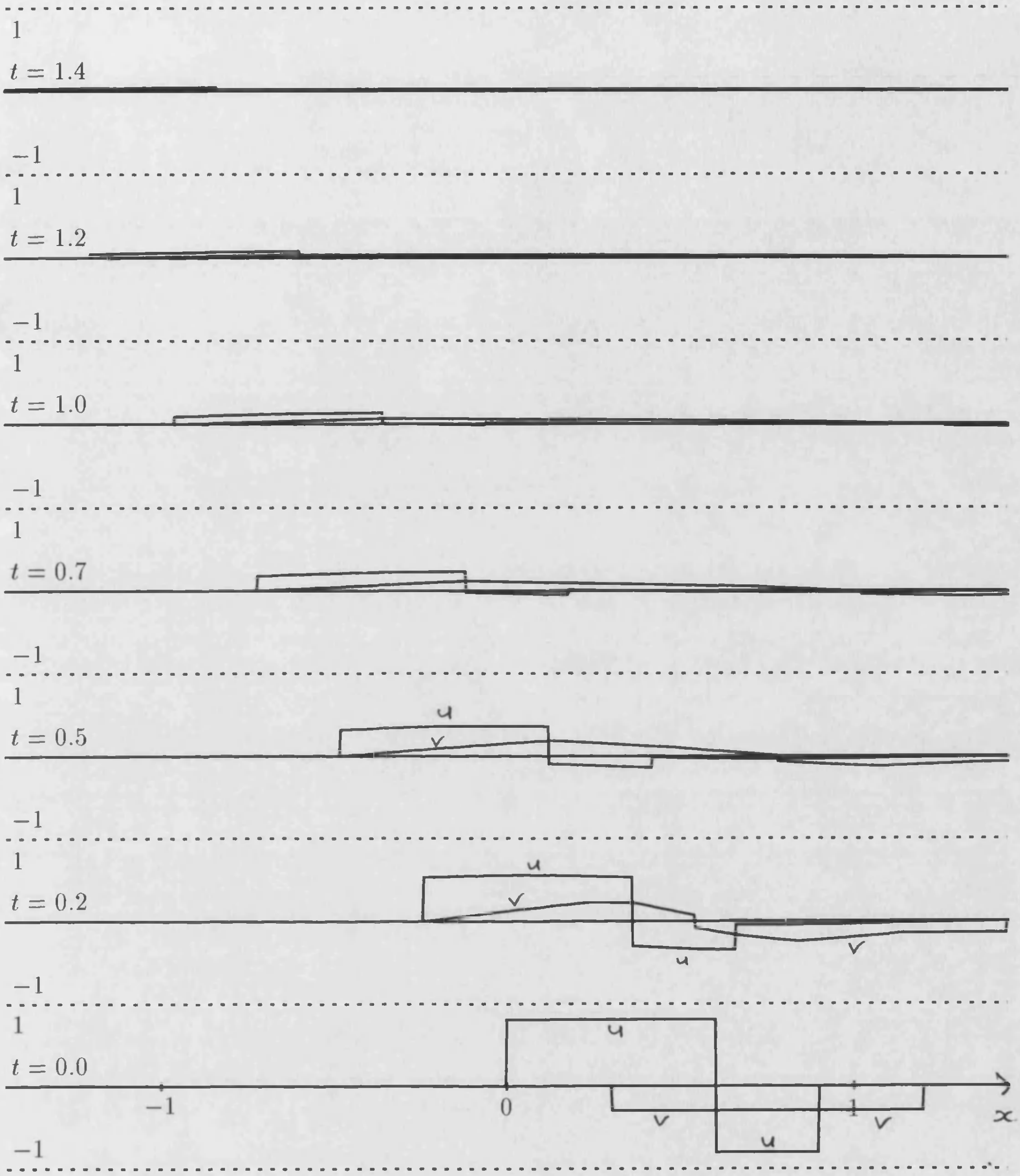
$$\frac{\partial v}{\partial t} = B\frac{\partial v}{\partial x} + (1 + 2K)R\left(\left(\frac{1+K}{1+2K}\right)v^2 - v + \left(\frac{K}{1+2K}\right)\right) + \theta Q v.$$

This transformed system has steady states $\left(\frac{K}{1+K}, \frac{K}{1+K}\right)$ and $(1, 1)$ (these are just the steady states of the original system in the new co-ordinates of course), while presence of further states will depend on the parameters.

Lemma 3.6 (with $K_1 = -K, K_2 = 1$) corresponds to the fact that the transformed system, if started with smooth data between 0 and 1, had a unique solution for all time, which stays between 0 and 1. This sets the stage nicely for a probabilistic representation of the transformed system.

The transformed system is represented by a model in which the breeding mechanism is changed slightly, while the motion and mutation remain the same. Now particles breed at rate $(1 + 2K)r_y$, but only give birth to a particle with probability $\left(\frac{1+K}{1+2K}\right)$, with probability $\left(\frac{K}{1+2K}\right)$ the particle dies leaving no further offspring. This corresponds to the change in the nonlinearity in the equation, we now have split the v^2 term into a squared part (which still represents splitting into 2 particles) and a constant (v^0) part (which represents death).

An example of the solution to the untransformed system with negative initial data is given in Figure 9-11. Note that the solution rapidly goes to zero.



$q_1 = 1.00, \quad q_2 = 2.00, \quad r_1 = 2.00, \quad r_2 = 1.00.$

Graphs of u and v for $t = 0.0(0.240)1.440$

IVP via Euler2 with $h = 0.0015$

Figure 9-11: Partly negative initial data analyzed with modified Euler method along the characteristics

Chapter 10

Conclusions and further work

It is clear that there is much scope for utilising the interplay of analysis and probability in differential equation problems. Some direct, natural extensions of the work here are:

- To study further n coupled equations, as in Chapter 7. The martingale methods of Chapter 6 should generalize in a straight-forward fashion and the work of Crooks [21] will extend the necessary algebraic results we need in Chapter 4.

- To incorporate death as well as birth for the particles. This has links with Wiener-Hopf theory and the transformation detailed in section 9.4. Introducing the possibility of the particles becoming extinct necessitates more care in the probabilistic analysis.

- To study more general breeding. As noted by McKean [49] and Dunbar [24] we can cope with more than just binary splitting, if the branching mechanism is simply generated then it can, for example, correspond to a polynomial non-linearity of the form

$$\sum_{k=0,k\neq 1}^{N} \alpha_k u^k - u \sum_{k=0,k\neq 1}^{N} \alpha_k,$$

  for $N \in \mathbb{N}$ and positive constants α_k (the model studied in this thesis has $\alpha_k = 0$ for $k \neq 2$; the α_0 term corresponds to death of a particle).

- To study probabilistically the travelling waves at critical speed, which should follow from the work of Neveu [53] via use of stopping lines.

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Colophon

- This thesis was written using LaTeX[43], which is derived from TeX[40]. The generic text font is 10 pt Computer Modern Roman but other fonts from the same font family are required.
- The bibliography follows the Harvard citation style.
- `C` is the low level language used in the programs used throughout this thesis.
- The diagrams in Chapter 5 were produced using `Maple`, all the other diagrams were produced directly in Postscript by specially written `C` code.
- All figures are stored in Postscript and labelled with PSfrag.
- `Maple`, LaTeXand all the `C` programs were run on SUN Sparcstations.

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